Photon Wave Mechanics and Spin-Orbit Interaction in Single Photons

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OUTLINE

1: How the Photon is usually taught

2: Elementary Theory of the Wave Function of a Photon

3: “Advanced” Theory of the Wave Function of a Photon

4: Spin-Orbit Interaction in a Single Photon

How the photon is usually taught:

Maxwell-Boltzmann vs. Bose-Einstein

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

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TIFF (Uncompressed) decompressor
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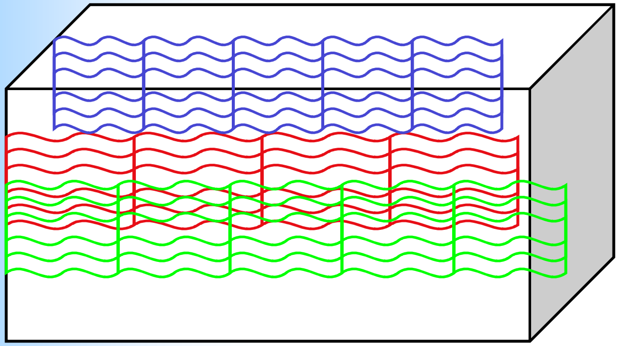
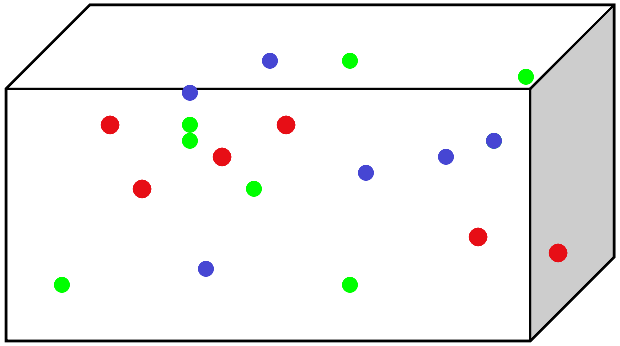
Light is made of EM waves.

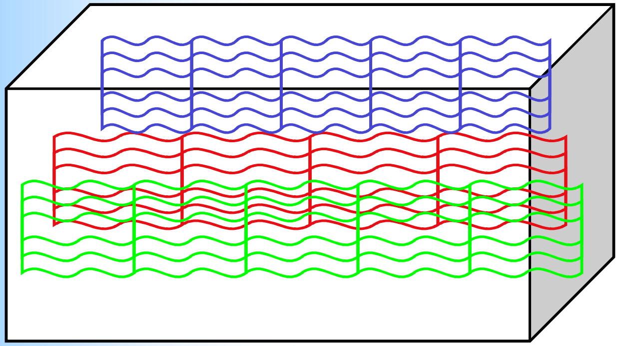


Light is made of corpuscles.



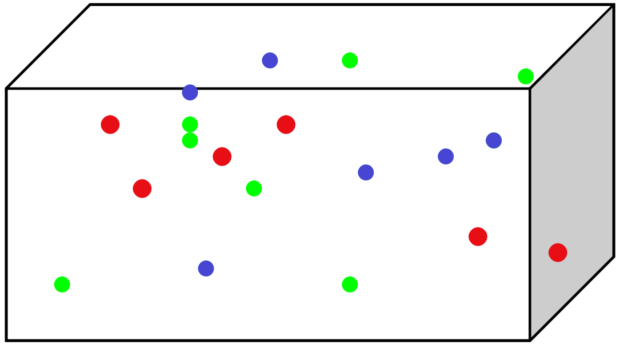
Maxwell-Boltzmann

Light is made of EM waves.
Modes are distinguishable.
M-B counting statistics applies.



Bose-Einstein

Light is made of corpuscles.
They are indistinguishable.
B-E counting statistics applies.



prediction → **Planck spectrum** ← prediction

inference | inference

EM fields as entities.
Photons as state description.(Dirac)

Not a change of basis. A change of viewpoint

Photons as entities.
Quantum field as “emergent.”

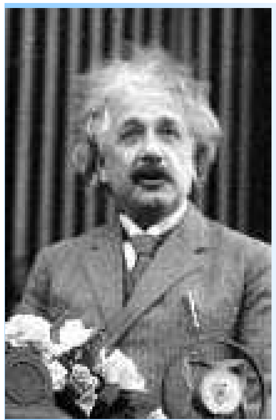
1. Photons are Bose particles with $E = h\nu$
2. Light is made of photons, but it also has wave properties, which are important when photons are flying through space, but not when they are detected.

Question: Must a photon be monochromatic?

An atom initially in an excited state decays spontaneously.

QuickTime™ and a
Animation decompressor
are needed to see this picture.

Question: If a photon can be in a fairly localized wave packet, what wave equation does this obey?



Al

Teaching Wave Mechanics for Particles - 1

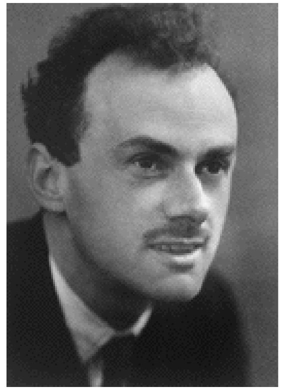
begin with Einstein's kinematic equation:

$$E = \sqrt{(m\,c^2)^2 + (c\,p)^2}$$

m=mass, p=momentum

(ignore polarization, spin, interactions)

$p = \hbar k$ (de Broglie) $\qquad E = \hbar\omega$ (Planck)



Paul

Dispersion relations --> Wave Equations in 1D

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electron: m>0, *v<<c*

$$E \simeq \cancel{mc^2} + \frac{p^2}{2m} + \ldots$$

$$\boxed{i\hbar \frac{\partial}{\partial t}\Psi \cong -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi}$$

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Electron Wave Equation

Erwin

photon: m=0, *v=c*

$$E^2 = c^2 p^2$$

$$\boxed{\cancel{\hbar^2}\frac{\partial^2}{\partial t^2}\Psi(x,t) = \cancel{\hbar^2}c^2\frac{\partial^2}{\partial x^2}\Psi(x,t)}$$

(*h* cancels out)

Photon Wave Equation

James Clerk

Teaching Wave Mechanics for Particles - 2

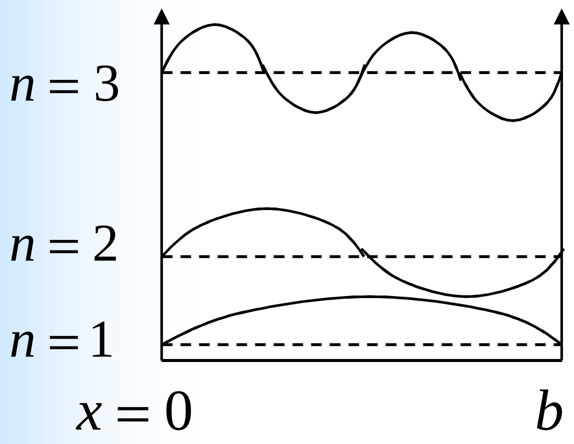
Particle in a Box

$$p = \hbar k \qquad E = \hbar\omega$$

$$\psi_n(x,t) = \sqrt{\frac{2}{b}}\sin(k_n x)\exp(-i\omega_n t) \qquad k_n = n\pi / b, \quad n = 1, 2, 3\ldots$$

Electron $\quad E = p^2/2m$

$$i\hbar\frac{\partial}{\partial t}\Psi \cong -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi$$

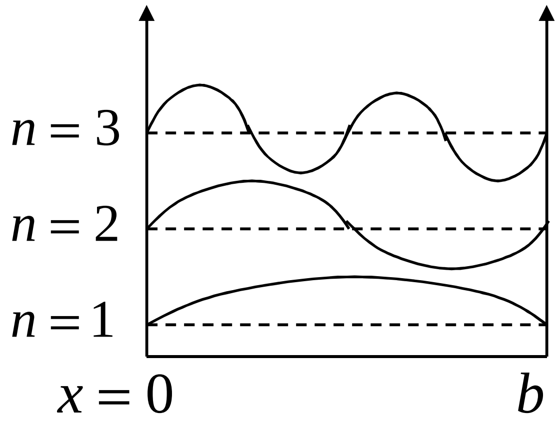


$$\omega_n = \frac{E_n}{\hbar} = \frac{(\hbar k_n)^2}{2m\hbar} = \left(\frac{n\pi}{b}\right)^2\frac{1}{2m\hbar}$$

(an electron “resonator”)

Photon $\quad E = cp$

$$\frac{\partial^2}{\partial t^2}\Psi(x,t) = c^2\frac{\partial^2}{\partial x^2}\Psi(x,t)$$



$$\omega_n = \frac{E_n}{\hbar} = \frac{c(\hbar k_n)}{\hbar} = n\frac{c\pi}{b}$$

(like a laser resonator)

Teaching Wave Mechanics for Particles - 3

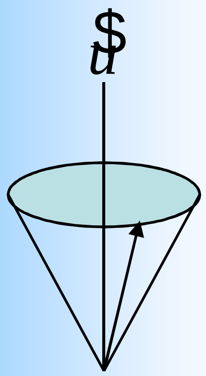
<u>Probability</u>: Born Interpretation

$|\Psi(x,t)|^2 =$ the probability density for finding the particle at position x at time t .

Electron	Photon
• Nonrelativistic: OK • Relativistic: Problematic - charge density not $\neq$ mass density	Always relativistic: Problematic, but OK for eigenstates of energy (or states with small spread in energy)

Teaching Wave Mechanics for Particles - 4

Spin: Just tack it on



Spin is described by two new quantum numbers, *s* and *m*

$$|\vec{S}| = \hbar\sqrt{s(s+1)} \qquad S_{\hat{u}} = \hbar m$$

Electron (s = 1/2)	Photon (s = 1)
$\lvert\vec{S}\rvert = \hbar\sqrt{(1/2)(1/2+1)}$	$\lvert\vec{S}\rvert = \hbar\sqrt{(1)(1+1)}$
$S_{\hat{u}} = \hbar m$ (any axis)	$S_{\hat{k}} = \hbar\sigma$ (propagation axis)
m = 1/2, -1/2 “spin projection”	σ = 1, -1 (not 0) “helicity”
$\Psi_{el} = \psi_{+\frac{1}{2}}(x,t)\chi_{+\frac{1}{2}} + \psi_{-\frac{1}{2}}(x,t)\chi_{-\frac{1}{2}}$	$\Psi_{ph} = \psi_{+1}(x,t)\chi_{+1} + \psi_{-1}(x,t)\chi_{-1}$
	(no worse than what we do to an electron)

So, What is a Photon?

1. **The <u>name</u> given to the n=1 states of the electromagnetic quantum field.**

---- or ----

2. **A fundamental quantum <u>particle</u>, through which the EM field emerges when many photons are present. (Like a nation emerging from an aggregate of many people.)**

Analogous statements hold for electrons.

Many details have been swept under the rug…

Derivation of Quantum Wave Equations

$$E = \sqrt{(mc^2)^2 + (cp)^2}$$ (Einstein)

(Planck) $i\hbar \frac{\partial}{\partial t} \Leftrightarrow E$

$\vec{p} \Leftrightarrow -i\hbar \vec{\nabla}$ (de Broglie)

$$i\hbar \frac{\partial}{\partial t}\Psi = \sqrt{(mc^2)^2 + c^2(-i\hbar\vec{\nabla})^2}\ \Psi$$

$\boxed{\text{electron}}$

$m \neq 0$

$s = 1/2$

$v \sim c$

Dirac Equation

$$i\hbar \frac{\partial}{\partial t}\Psi = cm\underline{\underline{\beta}}\Psi - i\hbar c(\underline{\underline{\vec{\alpha}}} \cdot \vec{\nabla})\Psi$$

$v \ll c$

4 components

Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t}\Psi^{(2)} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi^{(2)}$$

2 components

Require $\vec{\Psi}^* \vec{\Psi} =$ local number density

Derivation of Quantum Wave Equations

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$$i\hbar\frac{\partial}{\partial t}\Psi = \sqrt{(mc^2)^2 + c^2(-i\hbar\vec{\nabla})^2}\;\Psi$$

Can we do the same for a single photon?

electron

m≠O

s=1/2

v~c

Dirac Equation

$$i\hbar\frac{\partial}{\partial t}\Psi = cm\underline{\underline{\beta}}\Psi - i\hbar c(\underline{\underline{\vec{\alpha}}}\cdot\vec{\nabla})\Psi$$

v<<c

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Parallel treatment for photon:

m=O
s=1
v=c

electron

m≠O
s=1/2
v~c

Dirac Equation

$$i\hbar \frac{\partial}{\partial t}\Psi = cm\underline{\underline{\beta}}\Psi - i\hbar c(\underline{\underline{\vec{\alpha}}}\cdot\vec{\nabla})\Psi$$

4 components

v<<c

Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t}\Psi^{(2)} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi^{(2)}$$

2 components

$$\vec{\nabla}\cdot\vec{\Psi} = 0$$

Photon Wv. Equation

$$i\hbar \frac{\partial}{\partial t}\vec{\Psi} = \hbar c\sigma\vec{\nabla}\times\vec{\Psi}$$

3 components for each helicity

σ = helicity (±1) ($\hbar$ cancels)

Require $\vec{\Psi}^{*}\vec{\Psi}$ =
local energy density

Derivation of Photon Wave Equation

$$E = c\sqrt{\vec{p}\cdot\vec{p}}$$

photon, m=0, s=1,
3 components

$$E\,\psi(\vec{p},E) = c\sqrt{\vec{p}\cdot\vec{p}}\;\psi(\vec{p},E)$$

$$\vec{\Theta}\ \mathrm{B}\ \sqrt{\vec{p}\cdot\vec{p}} \stackrel{?}{=} i\,\vec{p}\times$$

$$\vec{\Theta}\vec{\Theta}\psi_T = i\,\vec{p}\times(i\,\vec{p}\times\psi_T) = (\vec{p}\cdot\vec{p})\psi_T - \cancel{\vec{p}(\vec{p}\cdot\psi_T)} = (\vec{p}\cdot\vec{p})\psi_T$$

momentum wave fn

$$\psi(\vec{p},E) = (\psi_x,\psi_y,\psi_z)$$

$$\psi = \psi_T + \psi_L$$

$$\vec{p}\times\psi_L = 0,\qquad \vec{p}\cdot\psi_T = 0$$

$$\boxed{E\,\psi_T(\vec{p},E) = c\,i\,\vec{p}\times\psi_T(\vec{p},E)}$$

Arbitrary weight fn.
$f(E)$

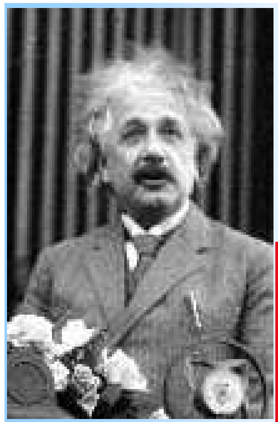
$$\vec{\psi}_T(\vec{r},t) \equiv \iint dE\,d^3p\;\delta(E - c|\vec{p}|)\exp(-iEt/\mathrm{h} + i\,\vec{p}\cdot\vec{r}/\mathrm{h})\,f(E)\,\psi_T(\vec{p},E)$$

$$\boxed{i\frac{\partial}{\partial t}\vec{\psi}_T(\vec{r},t) = c\,\vec{\nabla}\times\vec{\psi}_T(\vec{r},t)}$$

Photon Wave Equation

Require $\vec{\Psi}_T{}^*\,\vec{\Psi}_T$ =
local energy density. -->

$$f(E) = \sqrt{E}$$



$$E = \sqrt{(m c^2)^2 + (c\,p)^2}$$

(Einstein)

m=O

s=1

$$\vec{\nabla} \cdot \vec{\Psi} = 0$$

Photon Wv. Equation

$$i\hbar \frac{\partial}{\partial t} \vec{\Psi} = \hbar c \sigma \vec{\nabla} \times \vec{\Psi}$$

3 components

Require $\vec{\Psi}^* \vec{\Psi} =$
local energy density

$$\vec{\psi}(\vec{r},t) = \vec{\psi}_R + i\,\vec{\psi}_I$$

$$\begin{cases} \dfrac{\partial}{\partial t}\vec{\psi}_I = -c\,\vec{\nabla} \times \vec{\psi}_R \\ \dfrac{\partial}{\partial t}\vec{\psi}_R = c\,\vec{\nabla} \times \vec{\psi}_I \end{cases}$$

Compare to
Maxwell's Equations in Free

$$\begin{cases} \dfrac{\partial}{\partial t}\vec{B} = -c\,\vec{\nabla} \times \vec{E} \\ \dfrac{\partial}{\partial t}\vec{E} = c\,\vec{\nabla} \times \vec{B} \end{cases}$$

$$\begin{pmatrix} \vec{E} = \text{electric field} \\ \vec{B} = \text{magnetic field} \end{pmatrix}$$

For a single-photon field, the quantum wave function of the photon obeys the same wave equation as the complex electromagnetic field $E + \sigma i B$

helicity (spin, polarization): $\sigma = \pm 1$

$$i\frac{\partial}{\partial t}\vec{\Psi} = \sigma c\,\vec{\nabla}\times\vec{\Psi}$$

$$i\hbar\frac{\partial}{\partial t}\vec{\Psi} = \sigma\hbar c\,\vec{\nabla}\times\vec{\Psi} \equiv \hat{H}\vec{\Psi}$$

Maxwell, in 1862, discovered a fully relativistic, quantum mechanical theory of a single photon.

MODES $\Longleftrightarrow$ **STATES**

We can elevate the photon wave function to a quantum field, then the usual quantum field theory reappears. See the review:

"Photon wave functions, wave-packet quantization of light, and coherence theory,"

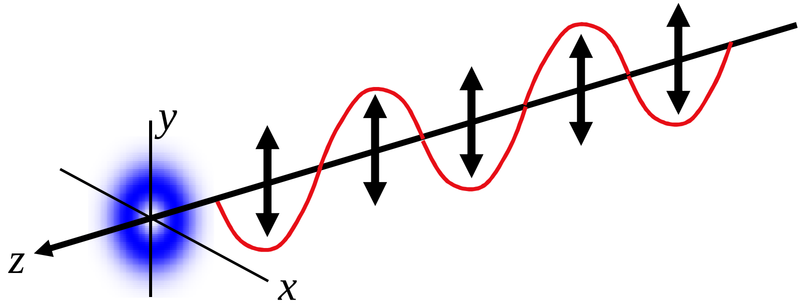
Brian J. Smith and M. R., New J. Phys. 9, 414 (2007)

There are subtleties:

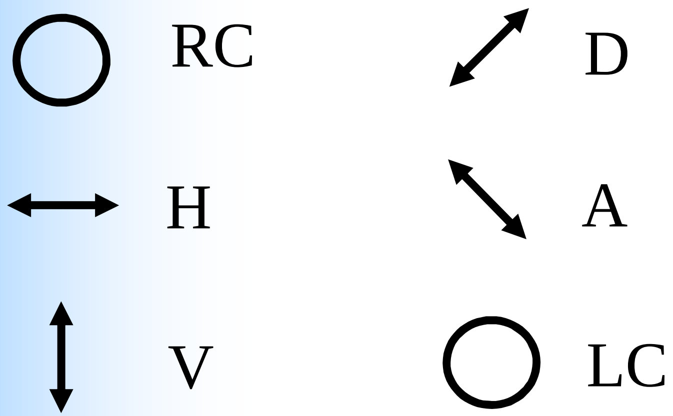
- $\vec{\Psi}_T^* \vec{\Psi}_T$ = energy density, not particle number density.
- Cannot localize a photon wave function to a point.
- The scalar (inner) product has an unusual form.
- There is NOT a Fourier-transform relation between momentum and position wave functions.

States (Modes) of Single Photons

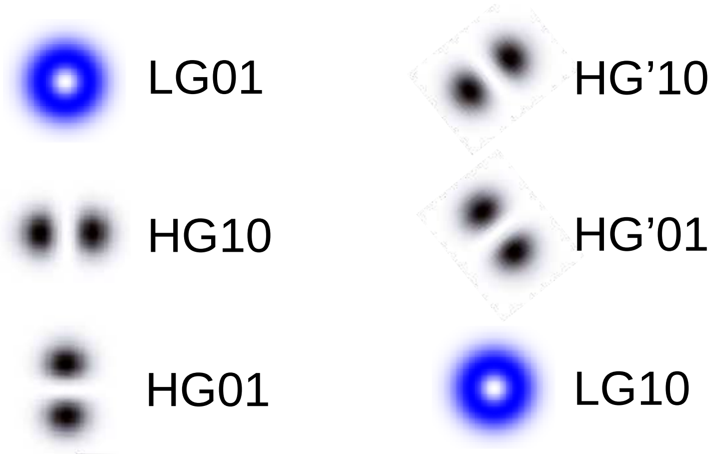
A photon has four degrees of freedom: momentum in *x*, *y*, and *z*; and *spin* (polarization).



<u>Polarization</u>



<u>Transverse Beam Shape</u>



Laguerre-Gauss and Hermite-Gauss Spatial Modes

To what extent is there a photon-electron analogy?

1. Spin Hall Effect for Electrons: opposite spin accumulation on opposing latteral surfaces of a current-carrying sample. Its origin is spin-orbit interaction.

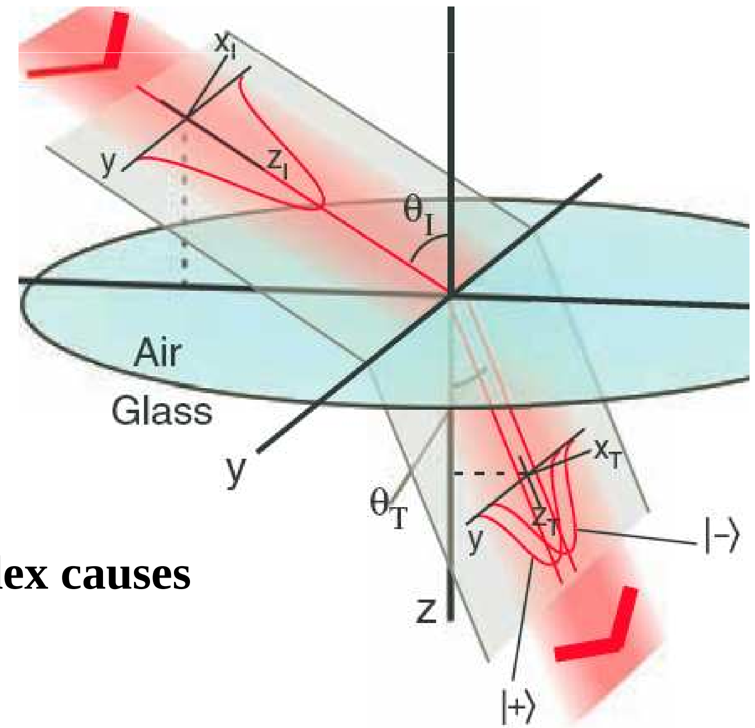
Dyakonov and Perel (1971) Sov. Phys. JETP Lett. 13, 467

Hirsch (1999) PRL 83, 1834

2. Spin Hall Effect for Light: spin-dependent displacement perpendicular to the refractive index gradient for photons passing through an air-glass interface.

M. Onoda, S. Murakami, N. Nagaosa, PRL 93, 083901 (2004)

Observed: Hosten, Kwiat Science 319 (2008)



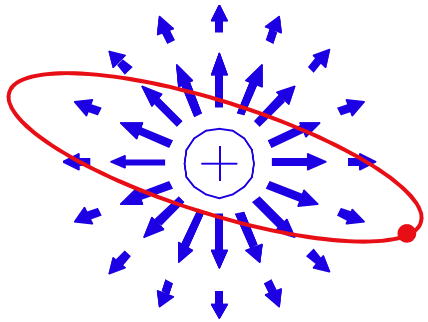
Inhomogeneity in refractive index causes SOI.

Spin-Orbit Interaction (SOI) in Spherical Potentials

• ELECTRON IN AN INHOMOGENOUS SPHERICAL ELECTRIC POTENTIAL (ATOM)

$$H' = -\frac{e^2}{2m^2c^2}\frac{1}{r^3}\mathbf{S}\cdot\mathbf{L}$$

$\boxed{\mathbf{S} = SAM}$ $\vec{r} \times \vec{p} = \boxed{\mathbf{L} = OAM}$

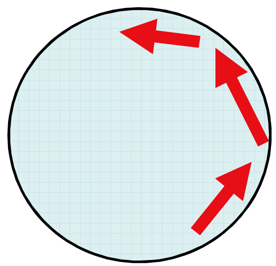


Coulomb potential

(atomic fine structure)

• PHOTON IN A DIELECTRIC SPHERE

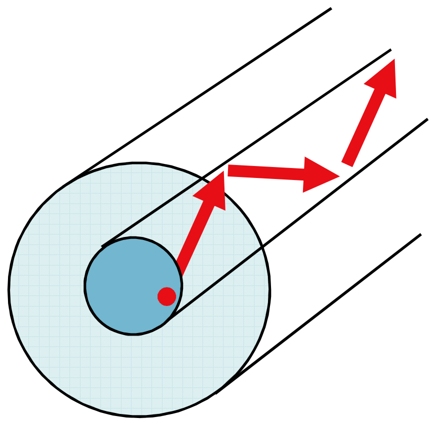
Polarization-dependent
mode-frequency shifts?



Spin-Orbit Interaction (SOI) in Cylindrical Potentials

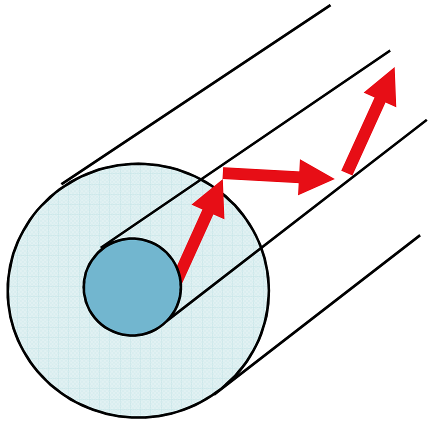
- ELECTRON IN AN CYLINDRCIAL WAVEGUIDE

Solve Dirac Equation for
the traveling-wave states.



- PHOTON IN A CYLINDRICAL OPTICAL FIBER

Solve Maxwell's Equations
for the modes and send a
single photon through.



ELECTRON in a CYLINDER STEP POTENTIAL

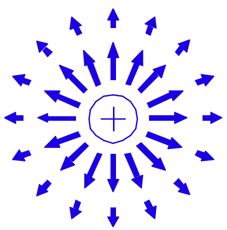
(C Leary, D Reeb, M Raymer, to appear NJP)

Dirac Equation --> Schrodinger Equation with SOI

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi + \underbrace{\frac{e}{m^2c^2}\mathbf{S}\cdot\left(\vec{\nabla}V\times\vec{p}\right)\Psi}_{\text{force on magnetic moment}} - \underbrace{\frac{e}{2m^2c^2}\mathbf{S}\cdot\left(\vec{\nabla}V\times\vec{p}\right)\Psi}_{\text{relativisitic Thomas factor}}$$

Recall Coulomb potential:

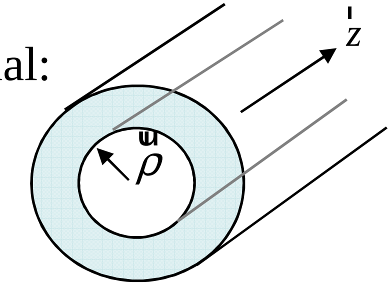
$$\vec{\nabla}V = -e\frac{\vec{r}}{r^3}$$



$$\mathbf{S}\cdot\left(\vec{\nabla}V\times\vec{p}\right) = -\frac{e^2}{r^3}\mathbf{S}\cdot\mathbf{L}$$

$\boxed{\mathbf{S} = SAM}$ $\quad\quad$ $\vec{r}\times\vec{p} = \boxed{\mathbf{L} = OAM}$

Cylindr. potential:



$$\vec{\nabla}V = \frac{\vec{\rho}}{\rho}\frac{\partial V}{\partial\rho}$$

$$\begin{aligned}\mathbf{S}\cdot\left(\vec{\nabla}V\times\vec{p}\right) &= \frac{1}{\rho}\frac{\partial V}{\partial\rho}\mathbf{S}\cdot\left(\vec{\rho}\times\vec{p}\right)\\ &= \frac{1}{\rho}\frac{\partial V}{\partial\rho}\mathbf{S}\cdot\left(\vec{\rho}\times\vec{p}_T\right) + \left[\,p_z\,term\,\right]\\ &= \frac{1}{\rho}\frac{\partial V}{\partial\rho}\underline{S_zL_z} + \left[\,p_z\,term\,\right]\end{aligned}$$

parallel or anti-parallel

ELECTRON in a CYLINDER STEP POTENTIAL

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi + H'; \qquad H' = \frac{e}{2m^2c^2}\frac{1}{\rho}\frac{\partial V}{\partial \rho}S_z L_z$$

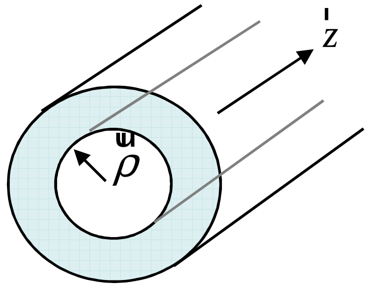
For fixed propagation constant (z-momentum),
perturbative Energy shift is: $\delta E = \langle \Psi | H' | \Psi \rangle$

$$\boxed{\frac{\partial V}{\partial \rho} = V_0\,\delta(\rho - a)}$$

where unperturbed states are

$$\Psi_{\sigma=+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} J_{m_l}(\kappa r)\, e^{i m_l \phi} e^{i(\beta z - \omega t)}$$

$$\Psi_{\sigma=-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} J_{m_l}(\kappa r)\, e^{i m_l \phi} e^{i(\beta z - \omega t)}$$



Then $$\delta E = \int \Psi_\sigma^{\dagger} H' \Psi_\sigma \propto \underline{(m_l\,\sigma)}\left(J_{m_l}(\kappa a)\right)^2$$

parallel AM

antiparallel AM

ELECTRON in a CYLINDER STEP POTENTIAL

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where unperturbed states are

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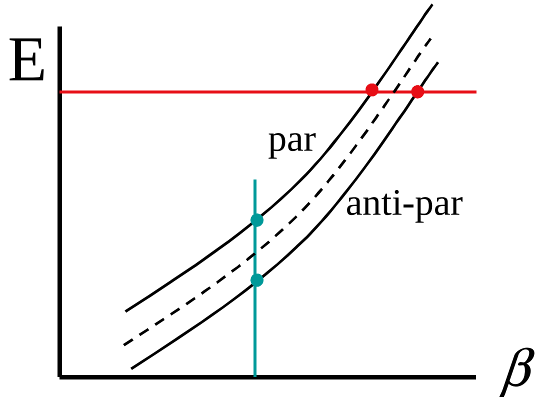
Then

$$\delta E = \int \Psi_\sigma^\dagger H' \Psi_\sigma \propto \underline{(m_l\,\sigma)}\left(J_{m_l}(\kappa a)\right)^2$$



- For a given energy, a <u>parallel-AM</u> electron state has a <u>smaller</u> z-propagation constant than that of an anti-parallel state.
- For a given z-propagation constant, a <u>parallel-AM</u> electron state has a <u>larger</u> energy than that of an anti-parallel state.
- Non-perturbative solution of Dirac equation gives same result.

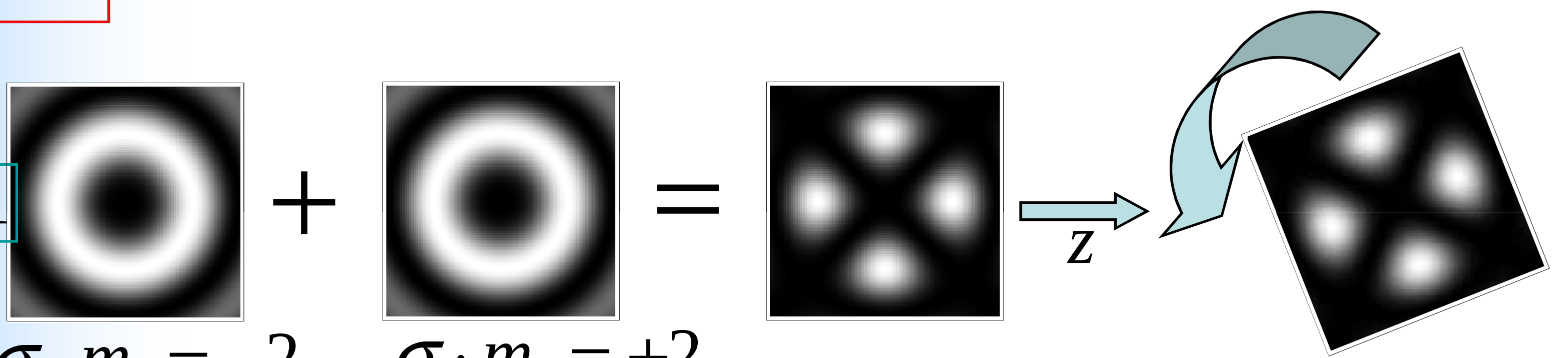
ELECTRON STATE ROTATION in a CYLINDER WAVEGUIDE

Superposition of degenerate positive-helicity states with opposite OAM:

$$e^{i m_l \phi + \beta_1 z} + e^{-i m_l \phi + \beta_2 z} \propto \cos\left[m_l \left(\phi + \sigma \frac{\beta_1 - \beta_2}{2} z \right) \right]$$

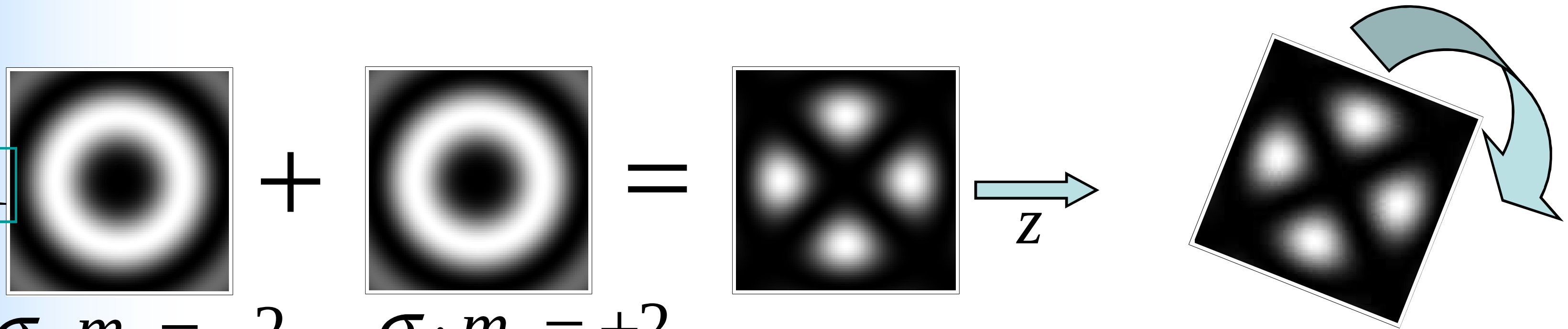
$\boxed{m_l = \pm 2}$

$\boxed{\sigma = -1}$



$\sigma \cdot m_l = -2$ $\qquad$ $\sigma \cdot m_l = +2$

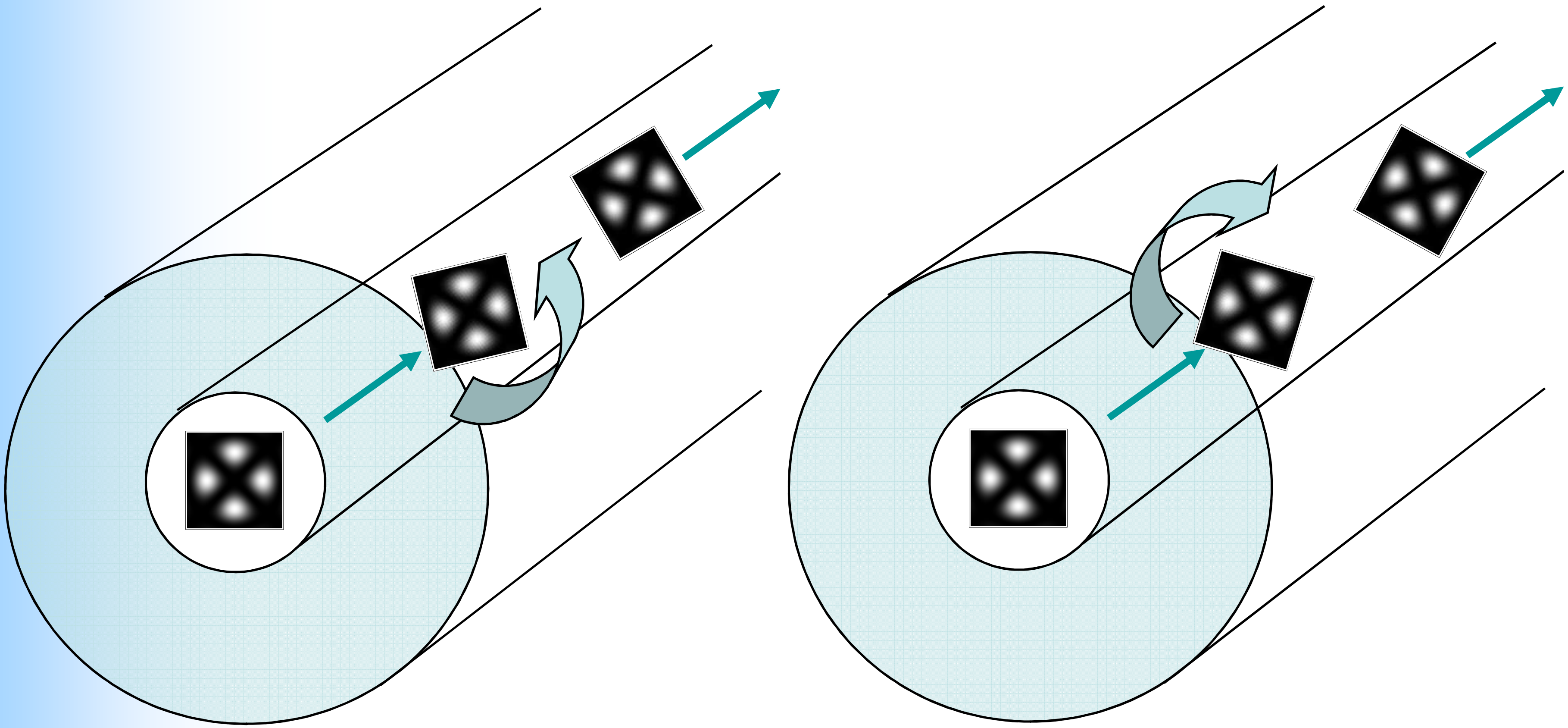
$\boxed{\sigma = +1}$



$\sigma \cdot m_l = -2$ $\qquad$ $\sigma \cdot m_l = +2$

ELECTRON STATE ROTATION in a CYLINDER WAVEGUIDE

Superposition of two degenerate positive-helicity states with opposite OAM: $m_l = +2, -2$

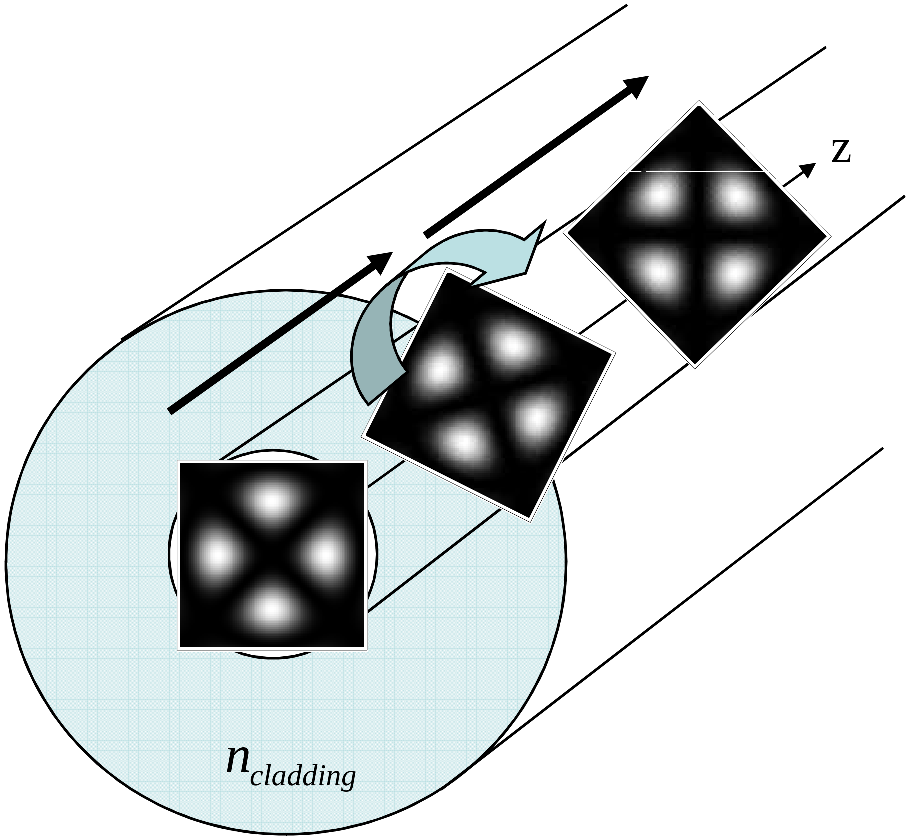


$\sigma = -1$

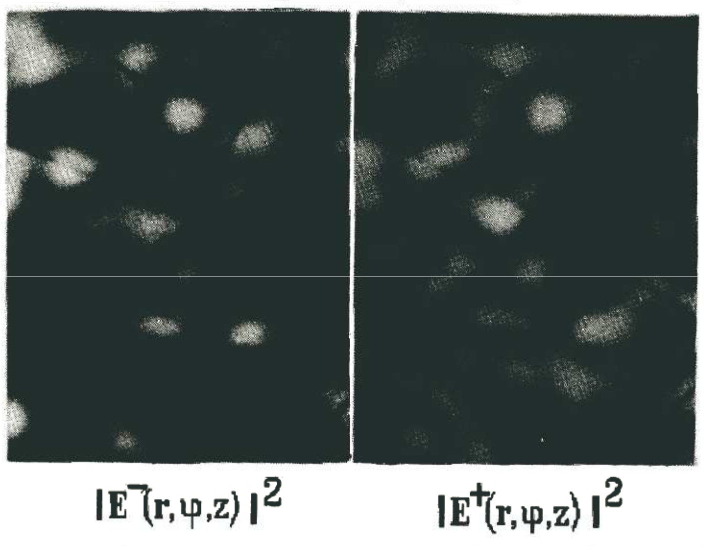
$\sigma = +1$

SPATIAL MODES ROTATION FOR LIGHT?

- Kapany and Burke first predicted polarization-dependent spatial mode rotation of optical modes in fiber. (1972)
- Did not explain in terms of SOI.



- Zel'dovich, Liberman (1990; PRA 46, 5199, 1992) first predicted optical SOI:
  - Treated a many-mode fiber with a parabolic index profile.
  - Predicted spatial mode rotation, due to SOI.
  - Observed rotation of speckle pattern, but not of single modes.



step-index 200 um

Dooghin et al PRA 1992

- Complementary to Rytov-Berry rotation of polarization by topological phase.
- See also works by A.V. Volyar.

Maxwell's Equations in an Inhomogeneous Medium, interpreted as the Quantum Wave Equation for a single photon

$$\frac{\partial D}{\partial t} = \nabla \times H, \quad \frac{\partial B}{\partial t} = -\nabla \times E$$

$$D = \varepsilon E, \quad H = B/\mu, \quad \nabla \cdot D = 0, \; \nabla \cdot B = 0$$

--> Photon Wave Equation:

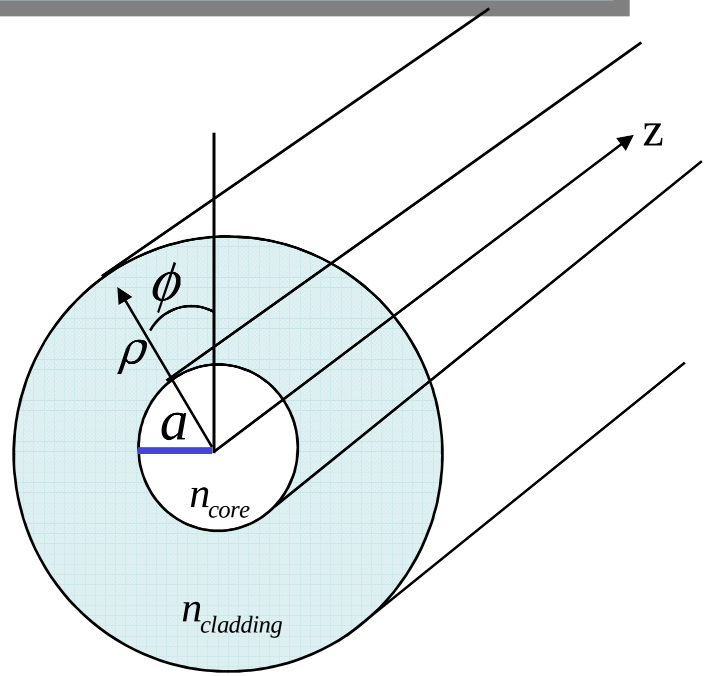
$$i\hbar \frac{\partial}{\partial t}\vec{\Psi} = \hbar c\,\vec{\nabla} \times \vec{\Psi} + \hbar c\,\vec{\nabla} N \times \vec{\Psi}$$

$$\left(\vec{\nabla} + \vec{\nabla} N\right) \cdot \vec{\Psi} = 0$$

(6 components)

I. Bialynicki-Birula (Prog.Opt.1996)

Perturbation Theory for Optical SOI in Step-Index Fiber (C Leary)



- Maxwell Wave Equation:

$$\nabla^2 E + \omega^2 \varepsilon(\rho) E + \nabla \left[\nabla \ln \varepsilon(\rho) \cdot E \right] = 0$$

$$E = \left(E_T + E_L \right) e^{i(\beta z - \omega t)}, \quad E_L << E_T$$

$$\Rightarrow \hat{H}_0 E_T + \hat{H}' E_T = \beta^2 E_T$$

$$\hat{H}_0 = \left(\nabla_T{}^2 + \omega^2 \varepsilon(\rho) \right)$$

$$\hat{H}' E_T = \nabla_T \left[\nabla_T \ln \varepsilon(\rho) \cdot E_T \right]$$

- Unperturbed eigenmodes have well defined components of spin σ and orbital angular momentum m_ℓ along *z* axis.

$$E(\rho, \phi, z) = \mathbf{e}_\sigma J_{m_\ell} \left(\kappa \rho \right) e^{i m_\ell \phi} e^{i(\beta z - \omega t)}$$

circular polarization vector

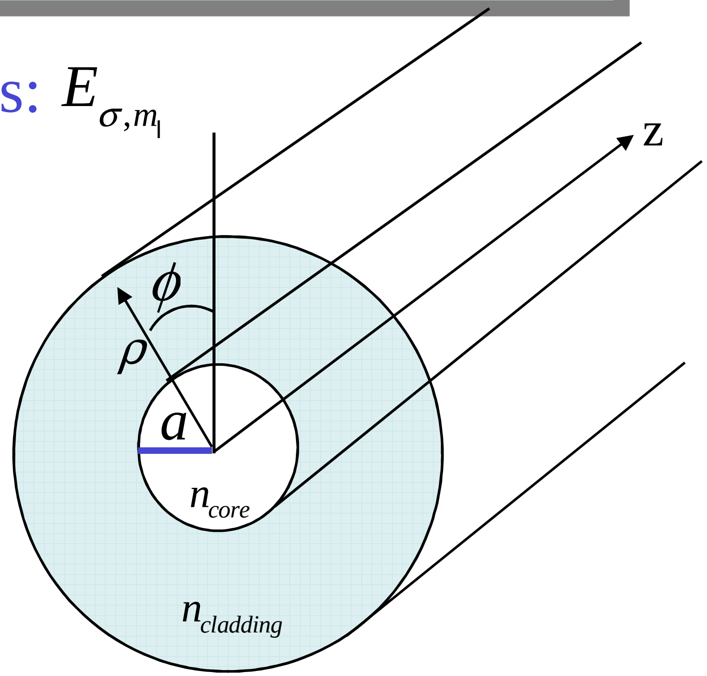
where $\mathbf{e}_\sigma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Perturbation Theory for Optical SOI in Step-Index Fiber (C Leary)

Perturbed Modes in Circular-Pol Basis States: E_{σ,m_l}

$$E_{+1,m_l} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} J_{m_l}\left(\kappa\rho\right) e^{i m_l \phi} e^{i\left([\beta + \underline{\delta\beta_{+1}}]z - \omega t\right)}$$

$$E_{-1,m_l} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} J_{m_l}\left(\kappa\rho\right) e^{i m_l \phi} e^{i\left([\beta + \underline{\delta\beta_{-1}}]z - \omega t\right)}$$



$$\delta\beta_\sigma \propto \left\langle E_{\sigma,m_l} \middle| \hat{H}' \middle| E_{\sigma,m_l} \right\rangle$$

$$\propto \left\langle E_{\sigma,m_l} \middle| \hat{\sigma}_3 \hat{L}_z \frac{1}{\rho}\frac{\partial \varepsilon}{\partial \rho} \middle| E_{\sigma,m_l} \right\rangle$$

$$\frac{\partial \varepsilon}{\partial \rho} \propto \delta(\rho - a)$$

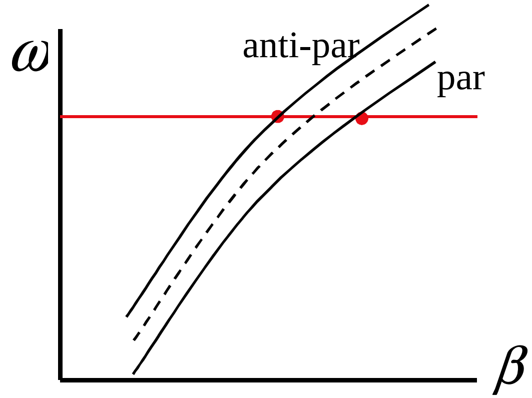
$$\boxed{\propto (\sigma m_l)\left(J_{m_l}(\kappa a)\right)^2}$$

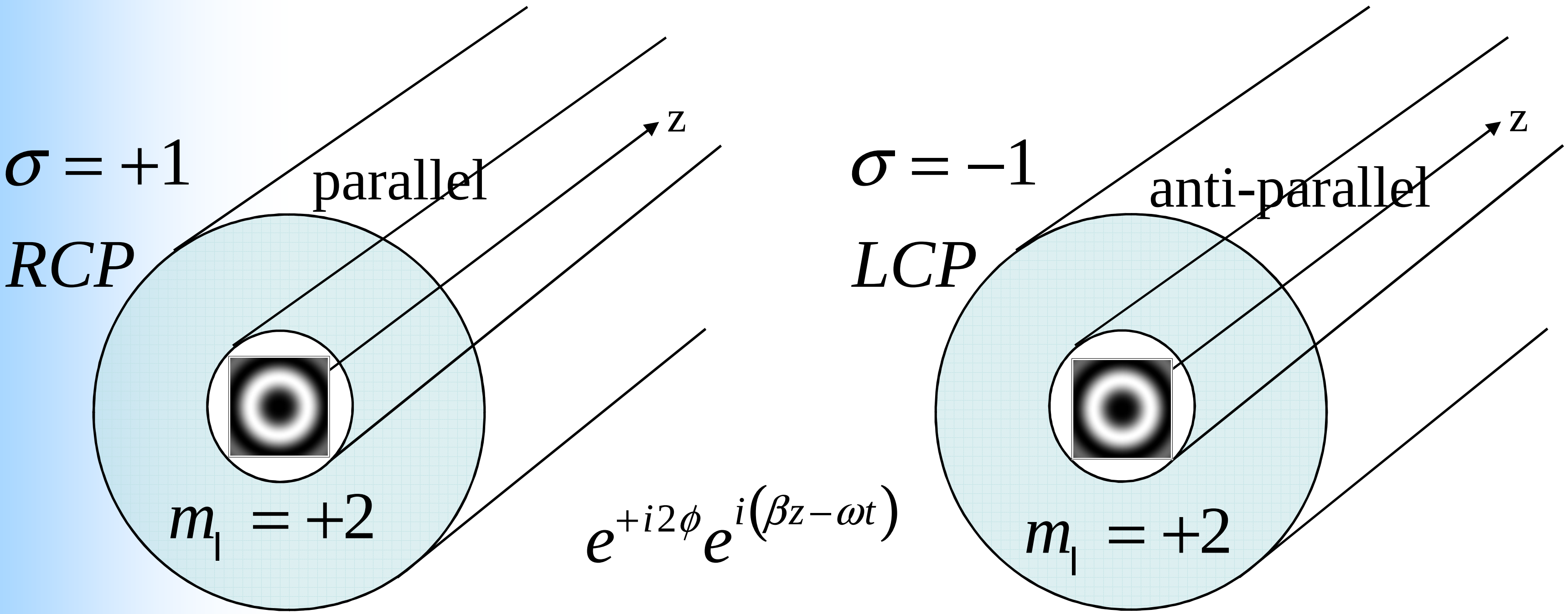
same as for electron!

$$\boxed{H'_{electron} \propto \hat{S}_z \hat{L}_z \frac{1}{\rho}\frac{\partial V}{\partial \rho}}$$

Nonperturbative Solutions for Optical SOI in Step-Index Fiber

propagation constant β is different when SAM and OAM are parallel or antiparallel (for fixed ω)



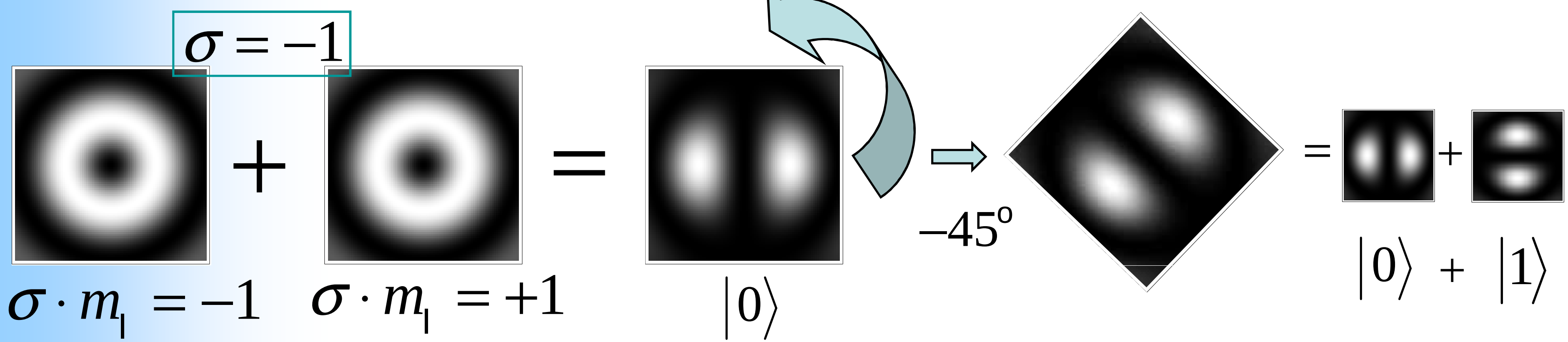


Photon spin angular momentum (SAM) and orbital angular momentum (OAM) can carry quantum information.

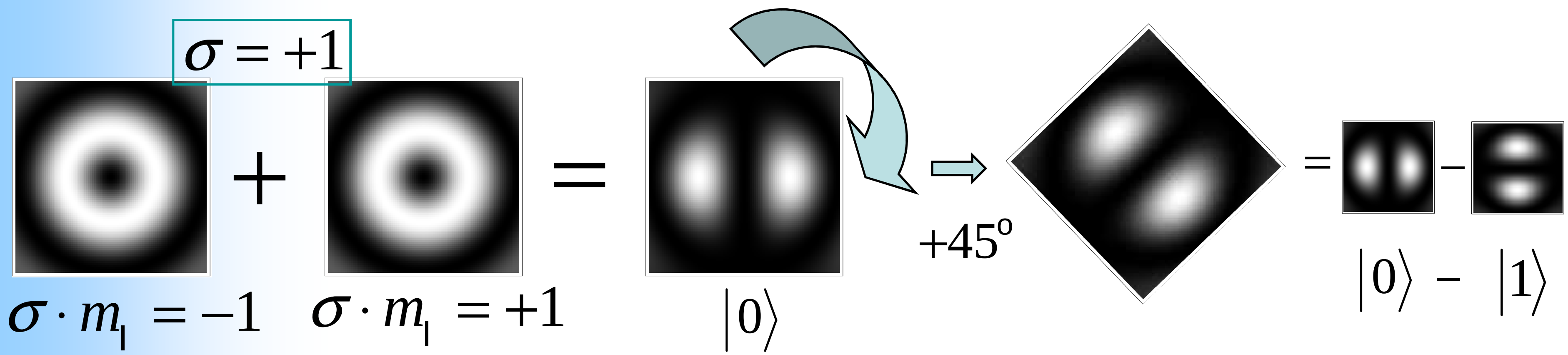
If photon SAM and OAM interact, then quantum gate interactions can perhaps be based on such interactions.

Single-Photon Spin-Controlled Hadamard Gate $|m_l| = 1$

$$e^{i m_l \phi + \beta_{+1} z} + e^{-i m_l \phi + \beta_{-1} z} \propto \cos\left[m_l \left(\phi + \sigma \frac{\beta_{+1} - \beta_{-1}}{2} z \right) \right]$$

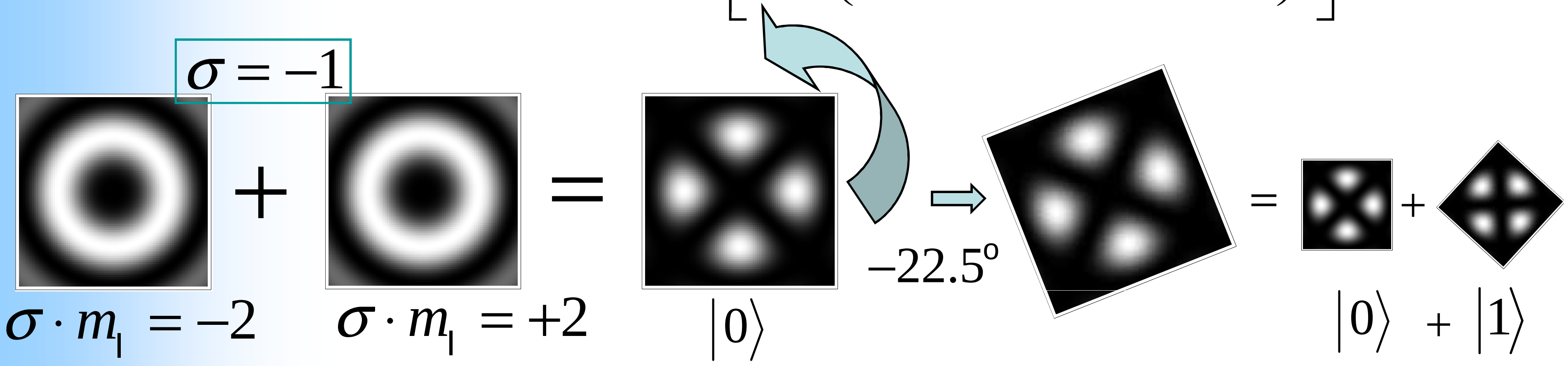


Flipping the photon spin (circular polarization) flips the direction of rotation of the superposition spatial mode.

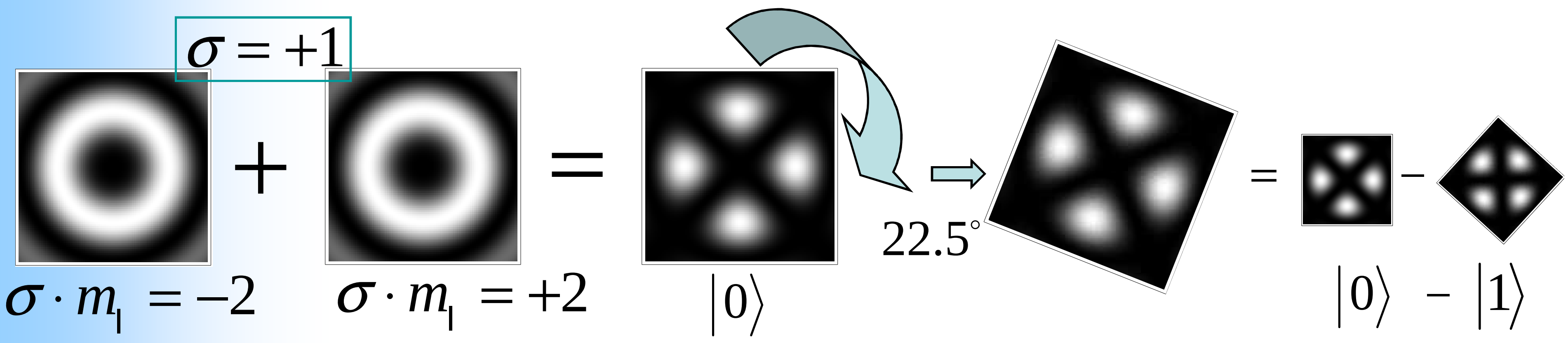


Single-Photon Spin-controlled Hadamard gate $\left| m_l \right| = 2$

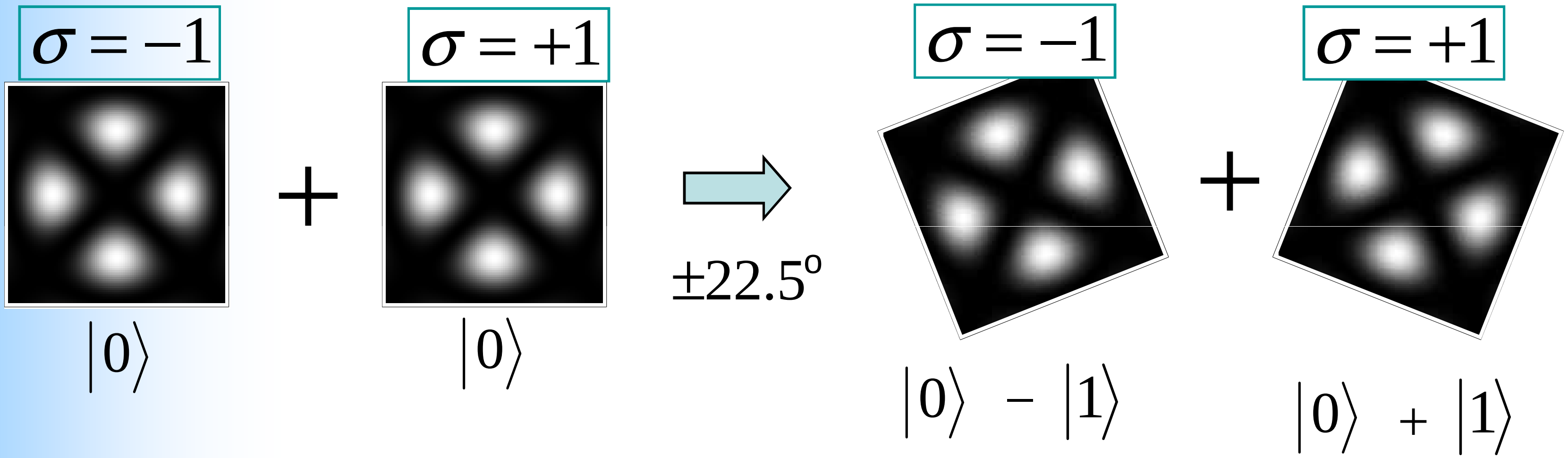
$$e^{i m_l \phi + \beta_{+1} z} + e^{-i m_l \phi + \beta_{-1} z} \propto \cos\left[m_l \left(\phi + \sigma \frac{\beta_{+1} - \beta_{-1}}{2} z \right) \right]$$



Flipping the photon spin (circular polarization) flips the direction of rotation of the superposition spatial mode.



SAM-OAM Entangling by Hadamard gate $\left|m_l\right| = 2$



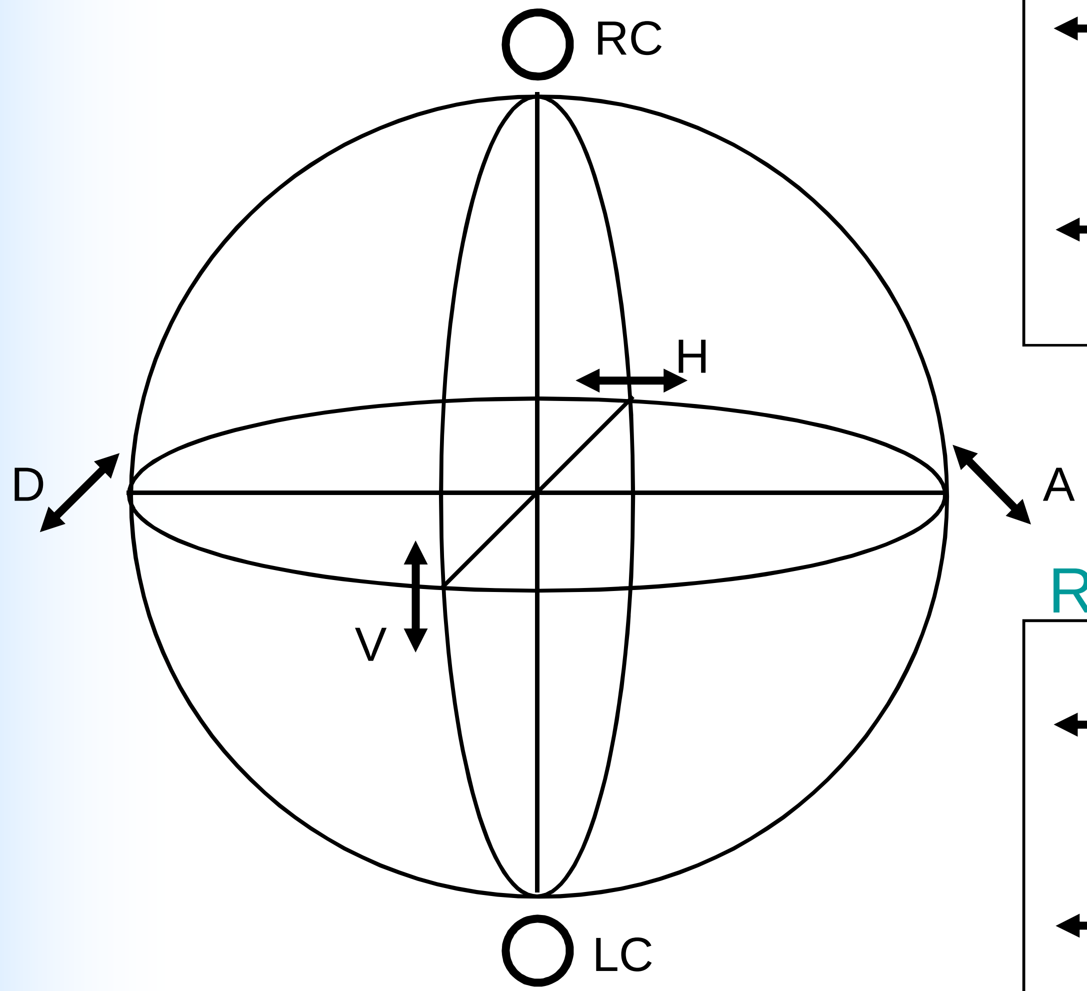
Summary: Spin-Orbit Interaction in Cylindrical Waveguides

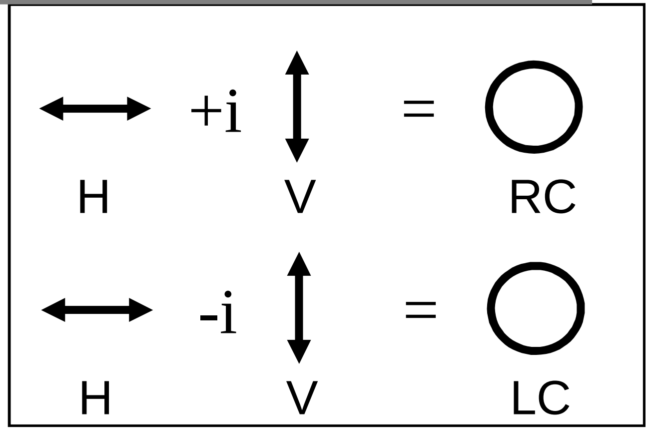
- Phase-velocity splitting proportional to σm_l.
- Parallel or anti-parallel SAM and OAM give rise to different propagation constants, for fixed frequency.
- Depends on total AM, $|m_j| = |m_l + \sigma|$.
- SOI-split states (modes) have a longitudinally varying relative phase difference, which creates rotation of superposition states (modes).
- Can be used to implement a single-photon spin-controlled spatial rotation, for entangling spin and spatial modes.
- Electron-photon analogy strengthens the photon-as-particle viewpoint.

Oregon Center for Optics

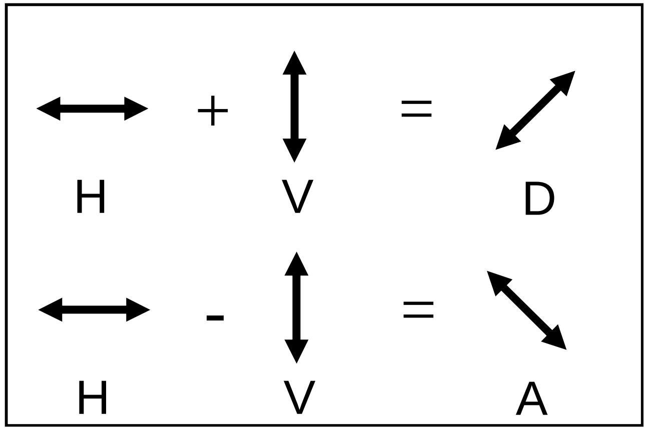
NSF

Poincare Sphere for Polarization

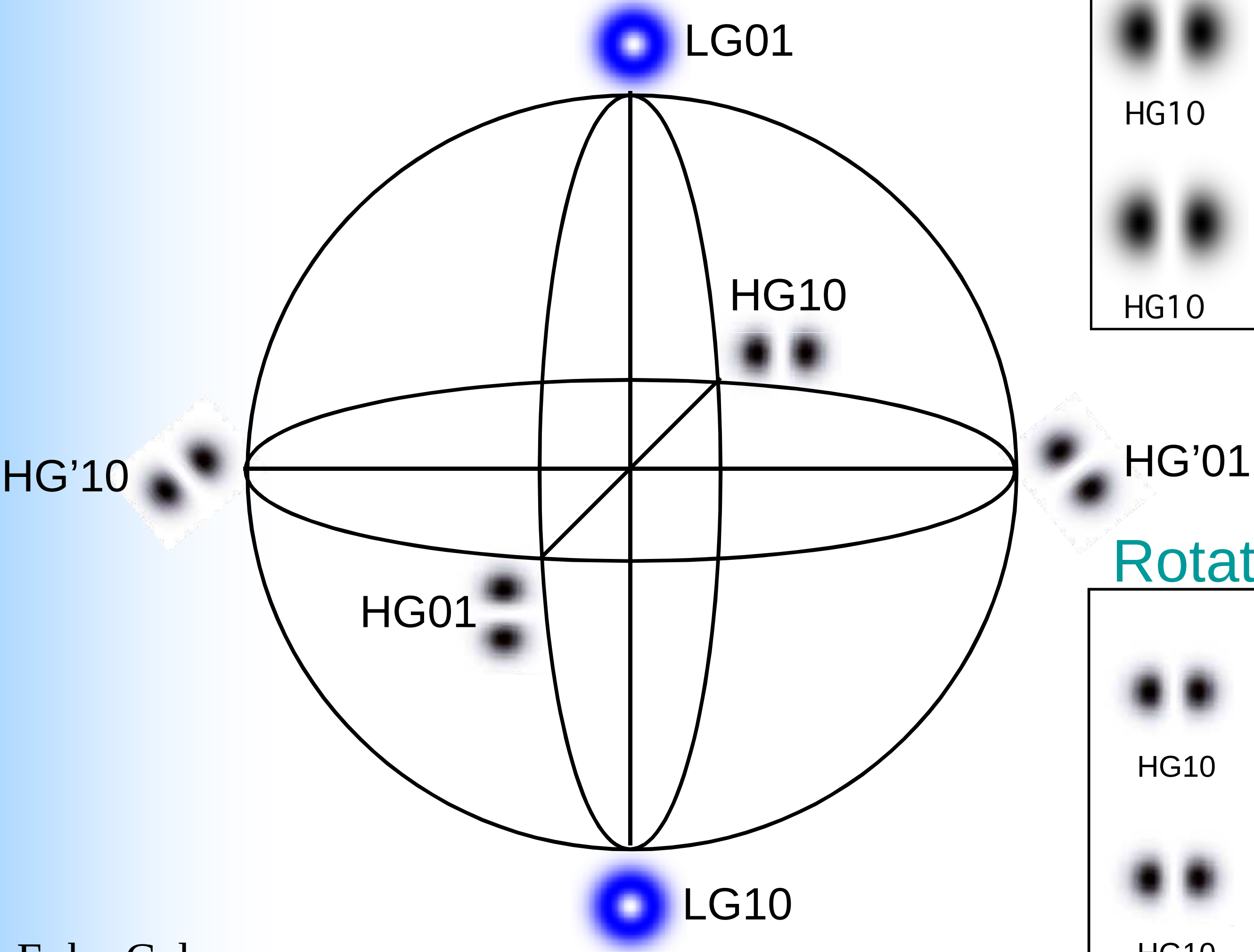


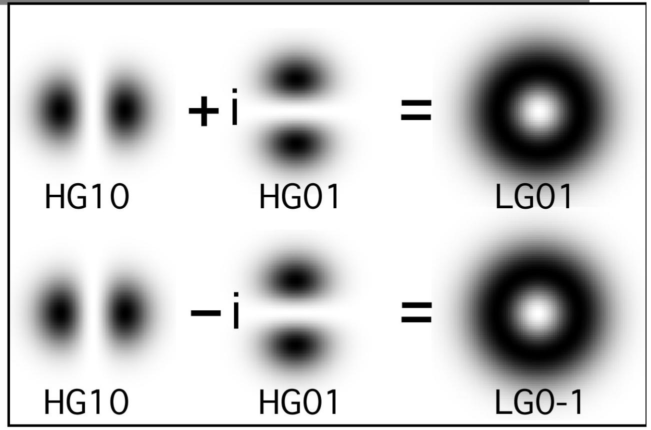


Rotation=Hadamard

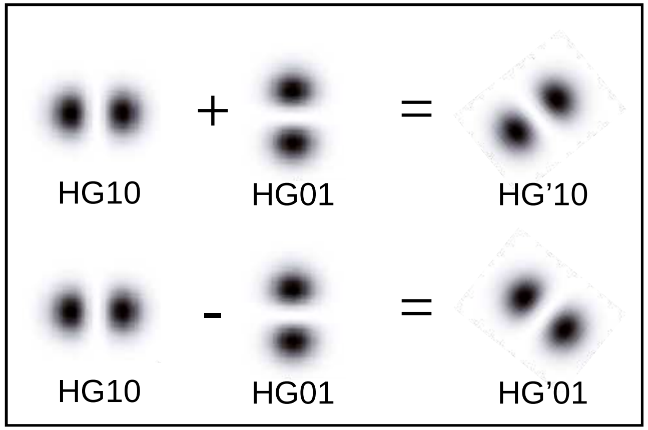


Poincare Sphere for L=1 Modes





Rotation=Hadamard



Van Enk; Galvez..

What are the proper Scalar Product and Normalization?

Bialynicki-Birula (1996)+refs.

- should be bilinear
- should be Lorentz invariant

$$\left(\vec{\Psi}_j \middle| \vec{\Psi}_m\right) \equiv \int \left[\int \frac{\vec{\Psi}_m^{*}(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^2} d^3r' \right] \vec{\Psi}_j(\vec{r})\, d^3r = \delta_{j,m}$$

Invariant,
Non-local

Norm: $$\left(\vec{\Psi} \middle| \vec{\Psi}\right) \equiv \int \left[\int \frac{\vec{\Psi}^{*}(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^2} d^3r' \right] \vec{\Psi}(\vec{r})\, d^3r = 1$$

No local particle density

(deal with it)

The mean Energy of the photon is:

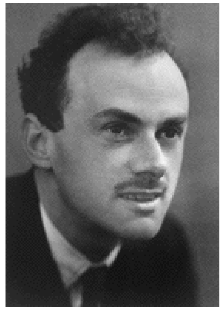
$$\left(\vec{\Psi} \middle| \vec{H} \middle| \vec{\Psi}\right) = \int \left[\int \frac{\vec{\Psi}^{*}(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^2} d^3r' \right] \hbar c \nabla \times \vec{\Psi}(\vec{r})\, d^3r = \int d^3r\, \vec{\Psi}(\vec{r})^{*}\, \vec{\Psi}(\vec{r}) = \langle H \rangle$$

$\left|\vec{\Psi}(\vec{r})\right|^2$ Is a local energy density.

Not invariant (OK)

$\vec{\Psi}(\vec{r})$ is the probability amplitude for localizing Energy, not particle position.

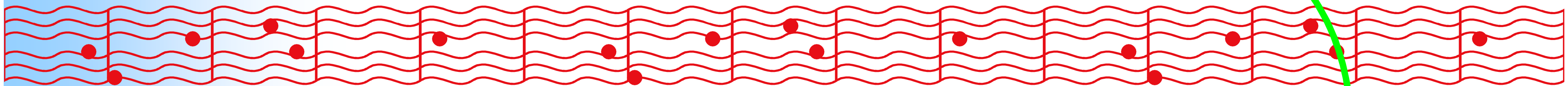
Quantum Field Theory: Dirac used Monochromatic Modes $(\vec{p} = \hbar \vec{k})$



$$\vec{E}^{(+)}(\vec{r},t) = \sum_{\sigma} \int d^3k \; \hat{a}(\vec{k},\sigma) \boxed{\sqrt{k} \; \boxed{\vec{\varepsilon}_{\sigma} \exp(i\vec{k}\cdot\vec{r} - i\omega t)}}$$

Bosonic operators:

$$\left[\hat{a}(\vec{k},\sigma),\, \hat{a}^{\dagger}(\vec{k}',\sigma')\right] = \delta(\vec{k},\vec{k}')\,\delta_{\sigma,\sigma'}$$



Quantum Field Theory using Temporal-Spatial (Wave-Packet) Modes

U. M. Titulaer and R. J. Glauber, Phys. Rev. 145, 1041 (1966)

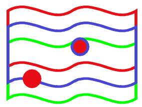
$$\vec{E}^{(+)}(\vec{r},t) = \sum_{j} \hat{b}_j \, \vec{v}_j(\vec{r},t) \quad , \quad \left[\hat{b}_j,\, \hat{b}_m{}^{\dagger}\right] = \delta_{j,m}$$

Unitary transformation:

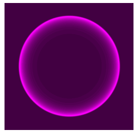
$$\hat{b}_j = \sum_{\sigma} \int d^3k \, \underline{R_j^{*}(\vec{k},\sigma)} \, \hat{a}(\vec{k},\sigma)$$

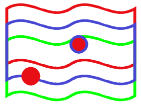
Non-Monochromatic modes (wave packets):

$$\vec{v}_j(\vec{r},t) = \sum_{\sigma} \int d^3k \, \underline{R_j(\vec{k},\sigma)} \, \underline{\sqrt{k}} \; \vec{\varepsilon}_{\sigma} \exp(i\vec{k}\cdot\vec{r} - i\omega t)$$



The T-G wave-packet modes are orthogonal under the same scalar product as are the photon wave functions





Photon Wave Functions:

$$\left(\vec{\Psi}_j \middle| \vec{\Psi}_m\right) \equiv \int \left[\int \frac{\vec{\Psi}_m^{*}(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^{2}} d^3r' \right] \vec{\Psi}_j(\vec{r})\, d^3r = \delta_{j,m}$$

L. Invariant, Non-local

Non-Monochromatic wave packet modes:

$$\vec{\mathrm{v}}_j(\vec{r},t) = \sum_{\lambda} \int d^3k\, R_j(\vec{k},\lambda) \underline{\sqrt{k}}\, \vec{u}_{\vec{k},\lambda}(\vec{r},t)$$

$$\left(\vec{\mathbf{v}}_j \middle| \vec{\mathbf{v}}_m\right) \equiv \int \left[\int \frac{\vec{\mathbf{v}}_m^{*}(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^{2}} d^3r' \right] \vec{\mathbf{v}}_j(\vec{r})\, d^3r = \delta_{j,m}$$

If we quantize the one-photon wave function, we obtain standard Dirac Quantum Field Theory

$$\boxed{i\frac{\partial}{\partial t}\vec{\psi}_j(\vec{r},t) = c\,\vec{\nabla}\times\vec{\psi}_j(\vec{r},t)}$$

$$\hat{\vec{\Psi}}(\vec{r},t) = \sum_j \hat{b}_j\,\vec{\psi}_j(\vec{r},t) \quad , \quad \left[\hat{b}_j, \hat{b}_m^{\dagger}\right] = \delta_{j,m}$$

Bosonic operators

energy e-states: $\int \vec{\psi}_m^{*}(\vec{r})\,\vec{\psi}_j(\vec{r})\,d^3r = \hbar\omega_j\,\delta_{j,m}$

$$\hat{\vec{\Psi}}(\vec{r},t) = \sum_j \sqrt{k_j}\,\hat{b}_j\left(\frac{\vec{\psi}_j(\vec{r},t)}{\sqrt{k_j}}\right)$$

$$\int\left(\frac{\vec{\psi}_j{}^{*}(\vec{r},t)}{\sqrt{k_j}}\right)\left(\frac{\vec{\psi}_m(\vec{r},t)}{\sqrt{k_m}}\right)d^3r = \delta_{j,m}$$

$$\boxed{\hat{\vec{\Psi}}(\vec{r},t) = \sum_j \sqrt{k_j}\,\hat{b}_j\,\vec{\varphi}_j(\vec{r},t)}$$

Dirac form