Classical–quantum mappings for unconventional phase transitions in geometrically frustrated systems

Stephen Powell & J. T. Chalker

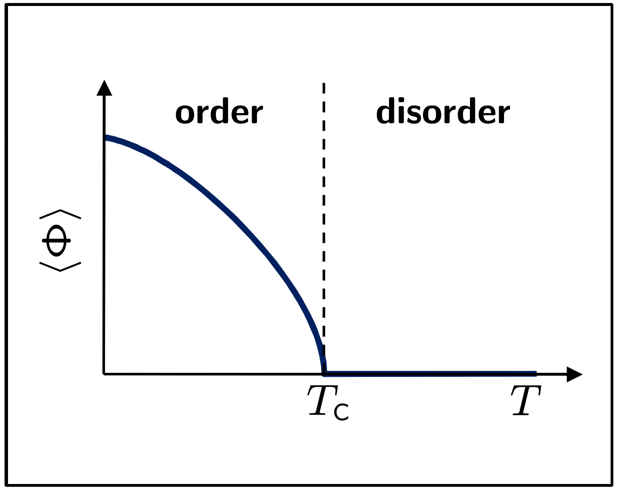
Kasteleyn transition of spin ice in a [100] magnetic field, Phys. Rev. B **78**, 024422 (2008)
Cubic lattice dimer model, Phys. Rev. Lett. **101**, 155702 (2008), arXiv:0907.1564 (to appear in PRB)

Summary

- Continuous classical transitions from Coulomb phases cannot be described by Landau-Ginzburg-Wilson theory
- Two examples in 3D:
  - Spin ice in a [100] magnetic field
  - Dimers on the cubic lattice
- Critical theories for these transitions can be found by mapping to 2+1D quantum models
  - For cubic dimers, the resulting model is an example of a non-LGW quantum transition

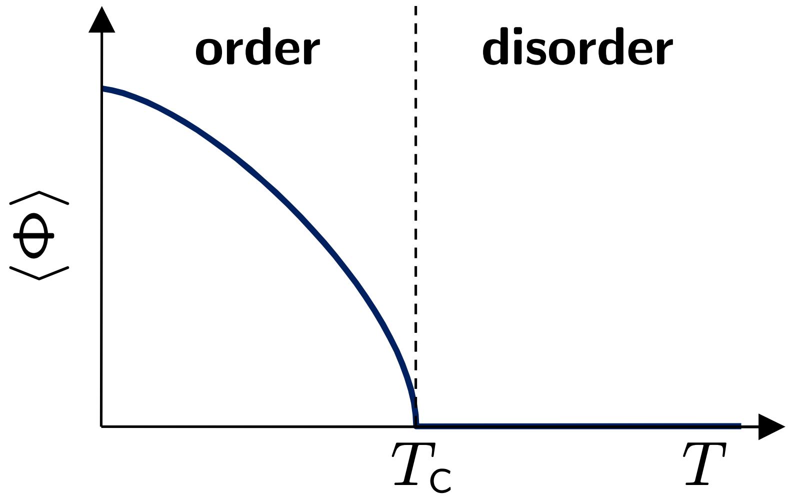
Outline

- LGW theory and Coulomb phases
  - Landau-Ginzburg-Wilson theory of phase transitions
  - Classical Coulomb/order transitions
- Classical–quantum mappings
  - Boson world-lines in spin ice
  - Quantum continuum theory
- Dimers on the cubic lattice
  - Unconventional phase transition
  - Mapping to quantum bosons
  - Critical theory: emergent SU(2) symmetry



Landau-Ginzburg-Wilson theory

Symmetry-breaking phase transition:



Effective action for fluctuations of Φ:

$$\mathcal{L} = |\boldsymbol{\nabla}\Phi|^2 + r|\Phi|^2 + u(|\Phi|^2)^2 + \cdots$$

(expansion in powers of Φ and derivatives,
including all terms with required symmetries)

Non-LGW quantum transitions

Spin-$\frac{1}{2}$ antiferromagnet with frustrating interactions

Néel order | valence-bond solid → frustration

$\boxed{\text{Senthil et al., Science } \mathbf{303}\text{, 1490 (2004)}}$

Mott-insulator / superfluid transition at fractional filling

CDW insulator | superfluid → hopping

$\boxed{\text{Balents et al., Phys. Rev. B } \mathbf{71}\text{, 144508 (2005)}}$

Classical Coulomb transitions

discrete degrees of freedom + strong local constraints

'ordered' phase	**Coulomb phase**
• short-ranged correlations • (symmetry breaking)	• power-law correlations • no symmetry breaking

non-LGW continuous — temperature T

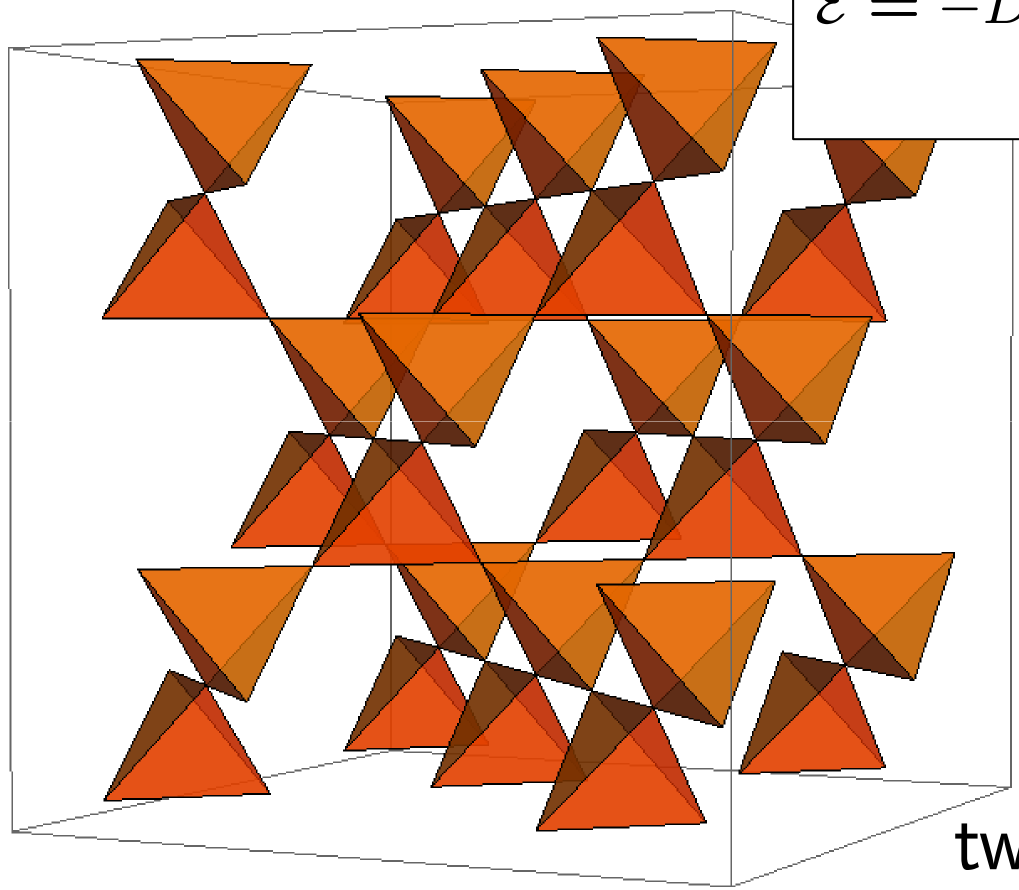
- Spin ice in a [100] magnetic field

  Jaubert et al., Phys. Rev. Lett. **100**, 067207 (2008)

- Classical dimer model on the cubic lattice

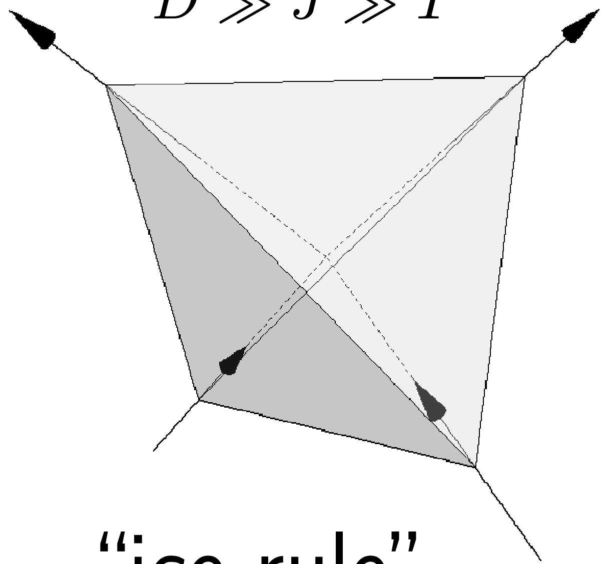
  Alet et al., Phys. Rev. Lett. **97**, 030403 (2006)

Spin ice



$$\mathcal{E} = -D \sum_i |\mathbf{e}_i \cdot \mathbf{S}_i|^2 - J \sum_{\boxtimes} |\mathbf{S}_{\boxtimes}|^2$$

$D \gg J \gg T$



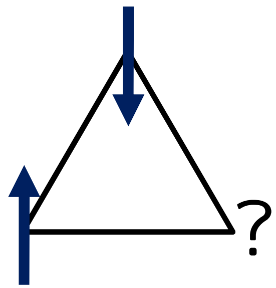
"ice rule"

two spins in, two spins out

Bramwell and Gingras, Science **294**, 1495 (2001)
Isakov et al., Phys. Rev. Lett. **95**, 217201 (2005)

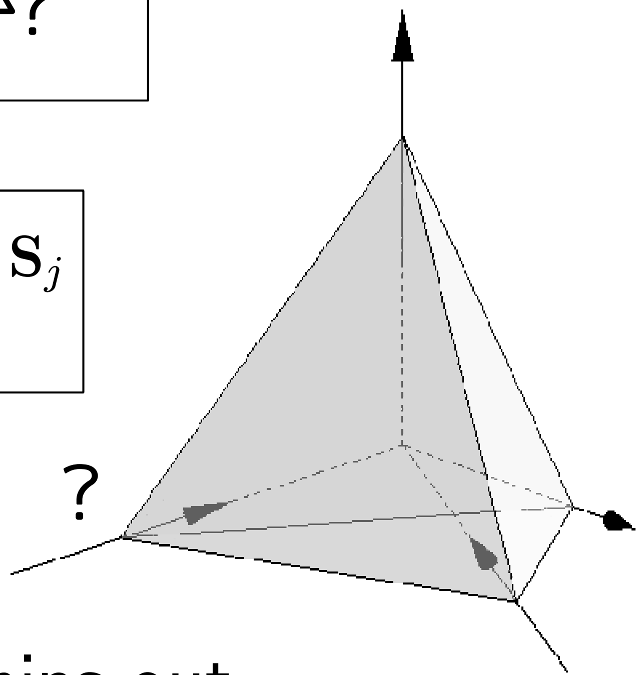
Frustration

$$\mathcal{E} = +J \sum_{\langle ij \rangle} S_i S_j$$



$$\mathcal{E} = -D \sum_i |\mathbf{e}_i \cdot \mathbf{S}_i|^2 - J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

$$D \gg J \gg T$$



"ice rule"

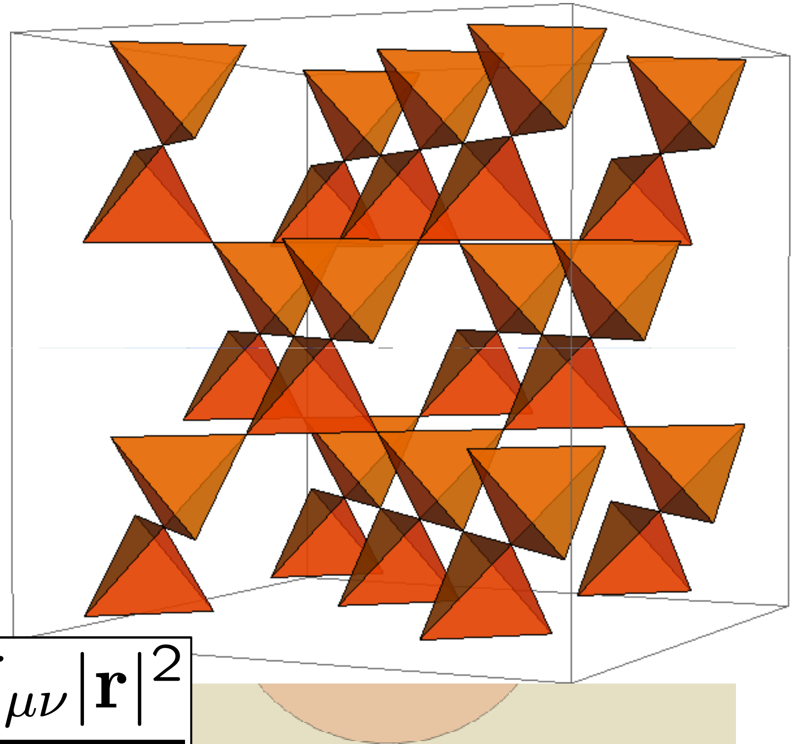
two spins in, two spins out

Moessner and Ramirez, Phys. Today **59**, 24 (2006)

Coulomb phase

$$\langle S_1^\mu S_2^\nu \rangle = \frac{1}{Z} \sum_{\text{configs}} S_1^\mu S_2^\nu$$

$D \gg J \gg T$



Coarse grain $\mathbf{S}_i$ to $\mathbf{S}(\mathbf{r})$

$$\boldsymbol{\nabla} \cdot \mathbf{S} = 0$$

Conjecture:

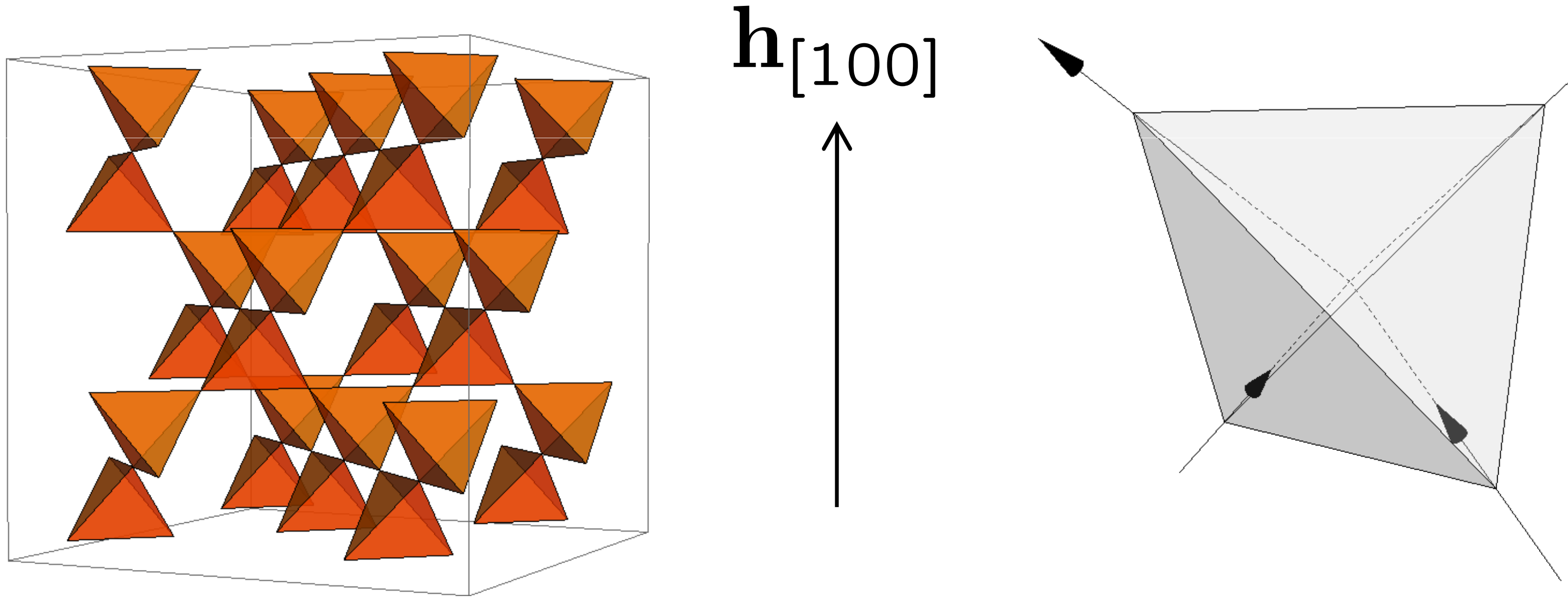
$$w \sim \mathrm{e}^{-\square|\mathbf{S}|^2}$$

$$\boxed{\langle S^\mu(\mathbf{r}) S^\nu(\mathbf{0}) \rangle \sim \frac{3 r_\mu r_\nu - \delta_{\mu\nu}|\mathbf{r}|^2}{|\mathbf{r}|^5}}$$

Henley, Phys. Rev. B **71**, 014424 (2005)
Isakov et al., Phys. Rev. Lett. **93**, 167204 (2004)

Spin ice in a [100] magnetic field

$$\boxed{\mathcal{E} = -D \sum_i |\mathbf{e}_i \cdot \mathbf{S}_i|^2 - J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - \sum_i \mathbf{h}_{[100]} \cdot \mathbf{S}_i}$$



Jaubert et al., Phys. Rev. Lett. **100**, 067207 (2008)
SP and J. T. Chalker, Phys. Rev. B 78, 024422 (2008)

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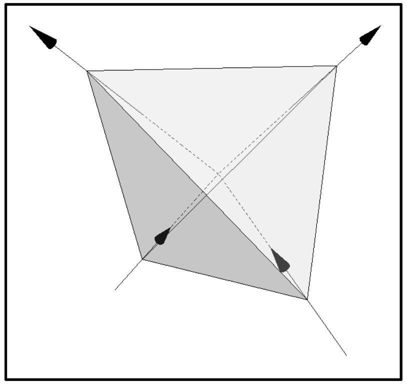
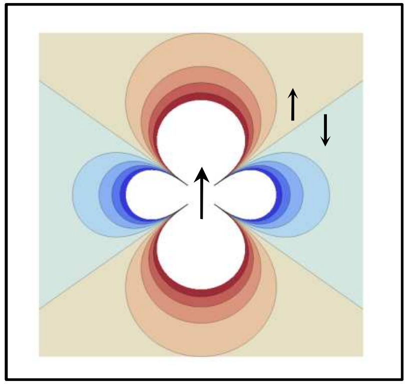
$$D \gg J \gg T, h$$





saturated

Coulomb

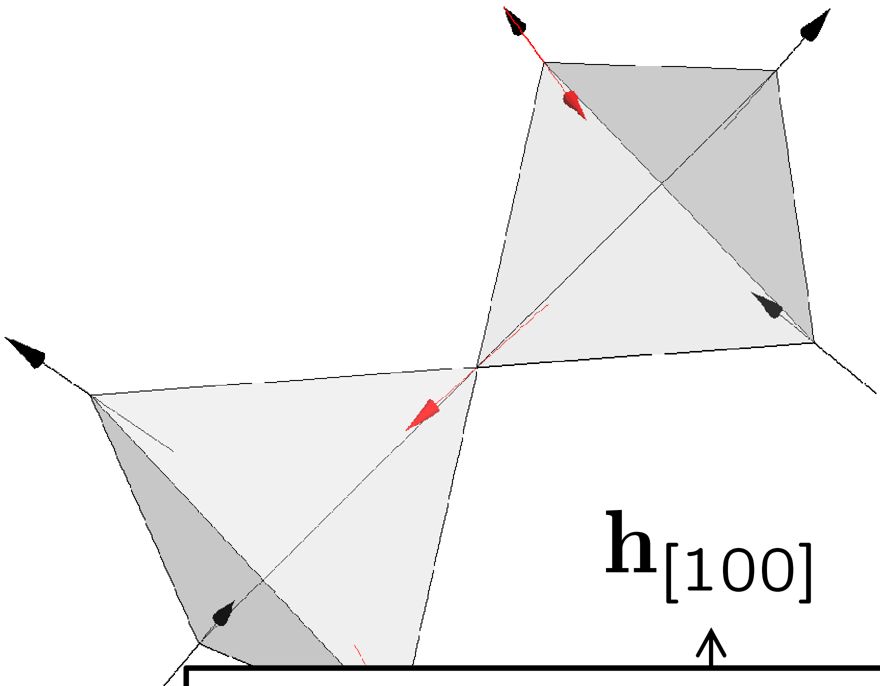
$T_\mathsf{c}(h)$

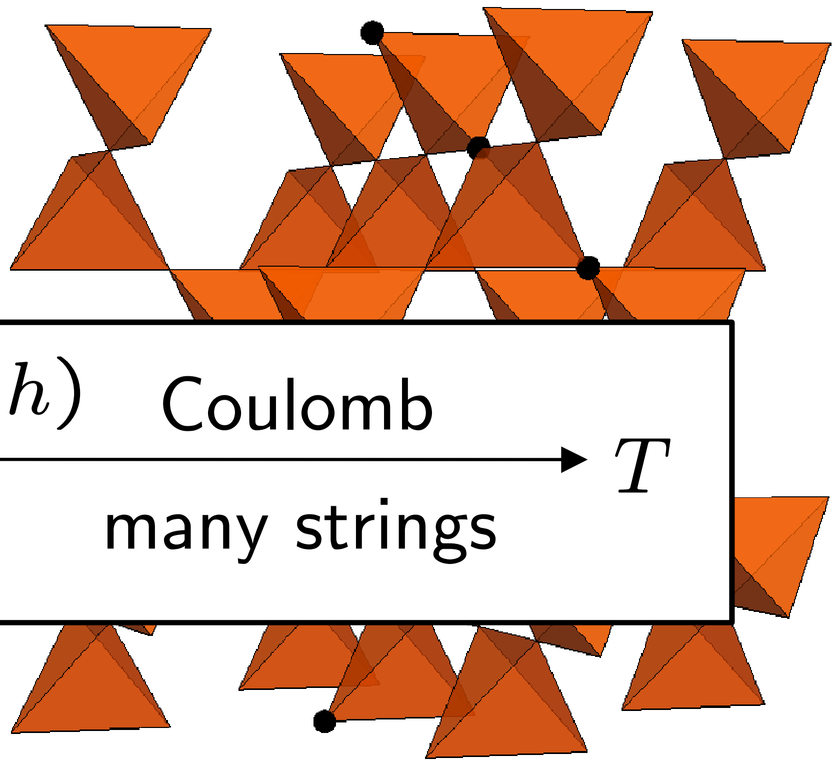
T

(fixed h)

String excitations

$$E \propto L_z h$$
$$S \propto L_z \log 2$$
$$\therefore T_{\mathsf{c}} \propto h$$





$\mathbf{h}_{[100]}$

saturated $T_{\mathsf{c}}(h)$ Coulomb

T

no strings many strings

$D \gg J \gg T, h$

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- Classical–quantum mappings
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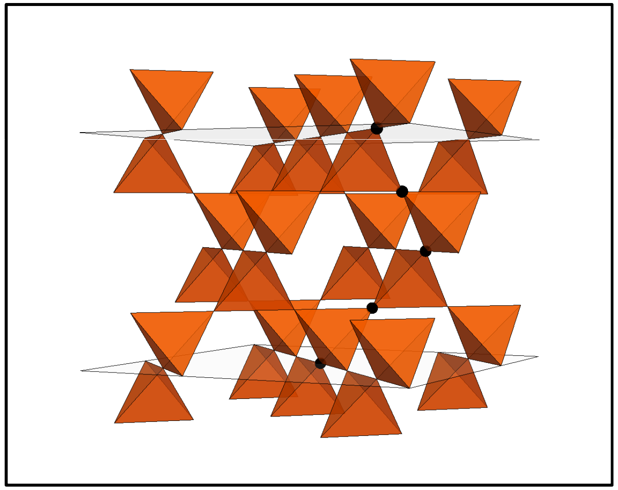
  - Quantum continuum theory

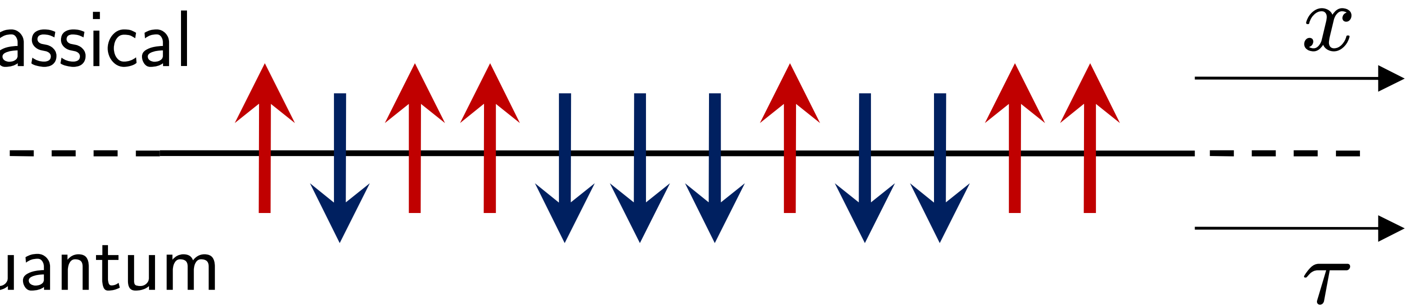


- Dimers on the cubic lattice
  - Unconventional phase transition
  - Mapping to quantum bosons
  - Critical theory: emergent SU(2) symmetry

Classical–quantum mapping

Example: classical Ising chain

1D classical

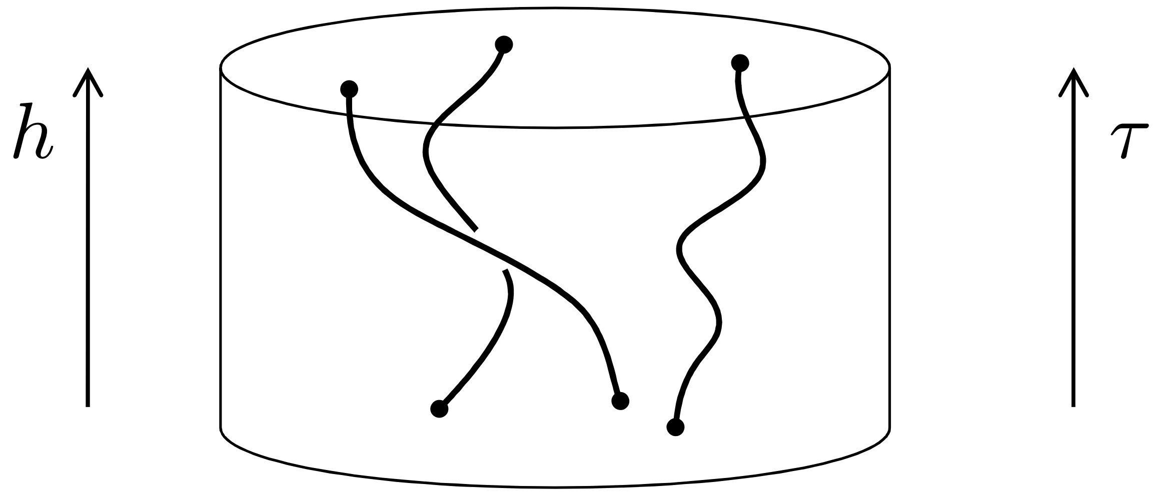


0D quantum

dD classical	$(d-1)$D quantum
system configuration	imaginary-time history
cross-section	quantum configuration
$\mathcal{Z} = \sum \mathrm{e}^{-\mathcal{E}/T}$	$\mathcal{Z} = \mathrm{Tr}\, \mathrm{e}^{-\beta \mathcal{H}}$

Sachdev, *Quantum Phase Transitions*, ch. 2

Mapping to quantum bosons



	saturated	Coulomb	
3D	no strings	many strings	T
2+1D	vacuum	superfluid	μ

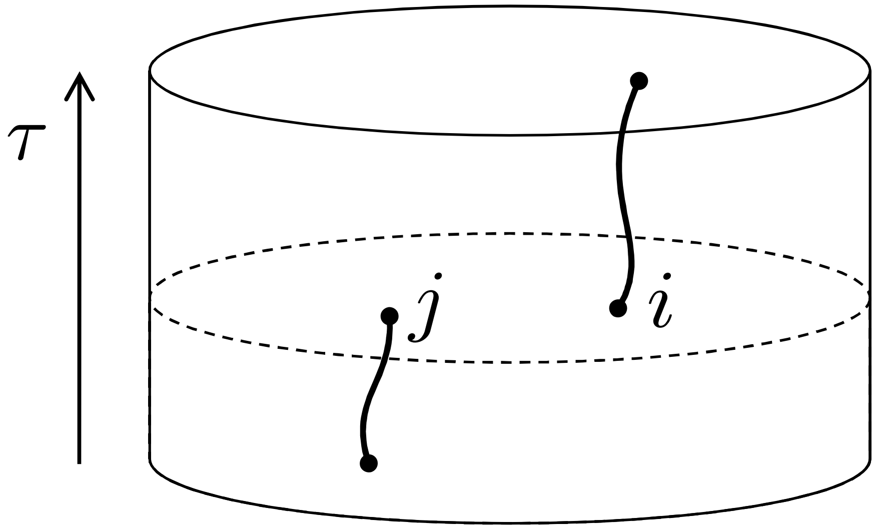
$$\mathcal{H} = -\mu \sum_i n_i - \sum_{ij} t_{ij} b_i^\dagger b_j + U \sum_i n_i (n_i - 1) + \cdots$$

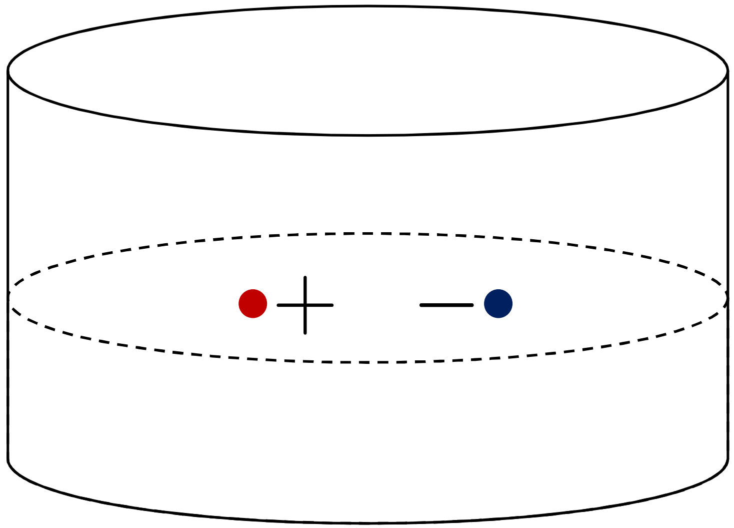
Sachdev, *Quantum Phase Transitions*, ch. 11

ODLRO* and deconfinement

*off-diagonal long-range order

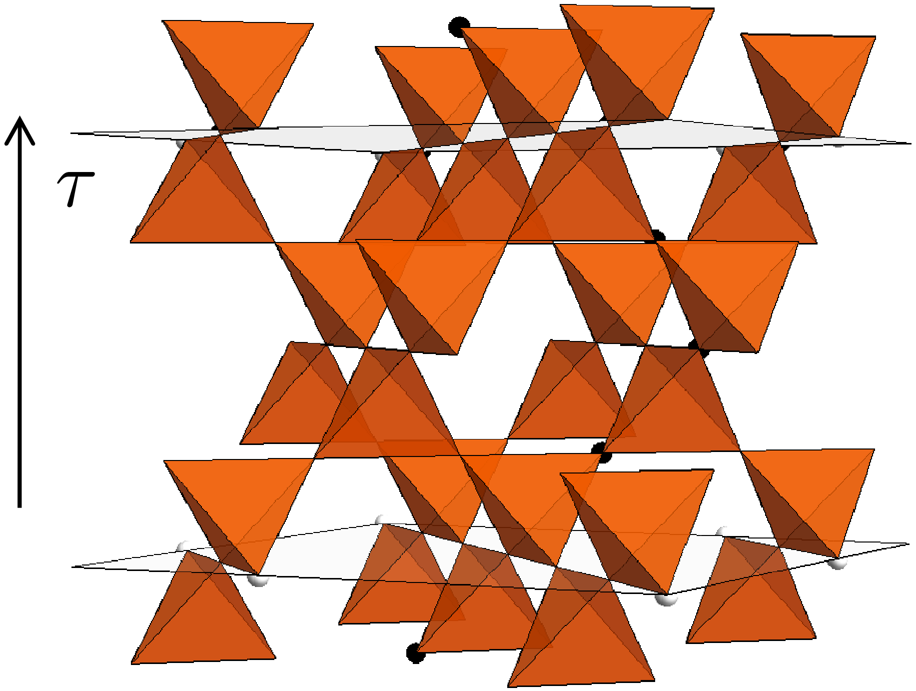
$$\langle b_i^\dagger b_j \rangle \to \text{const}$$

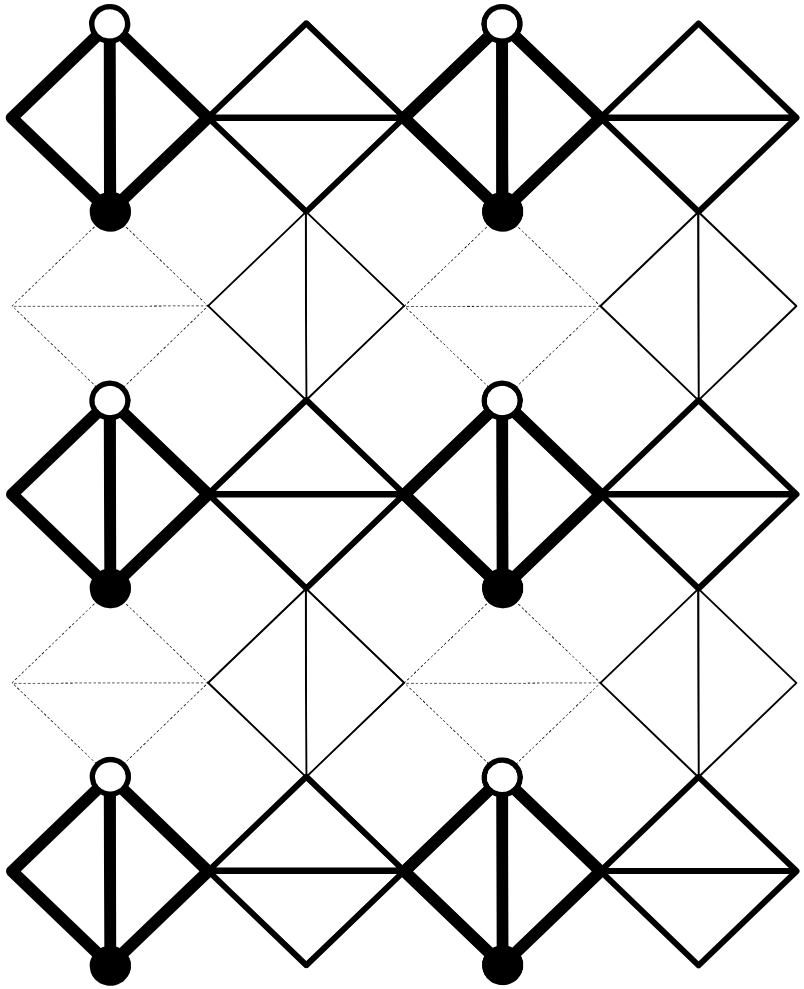




$$\langle b_i^\dagger b_j \rangle = \frac{\mathcal{Z}_{ij}}{\mathcal{Z}} \to \mathrm{e}^{-2\epsilon_\mathrm{m}/T}$$

Symmetries





SP and J. T. Chalker, Phys. Rev. B 78, 024422 (2008)

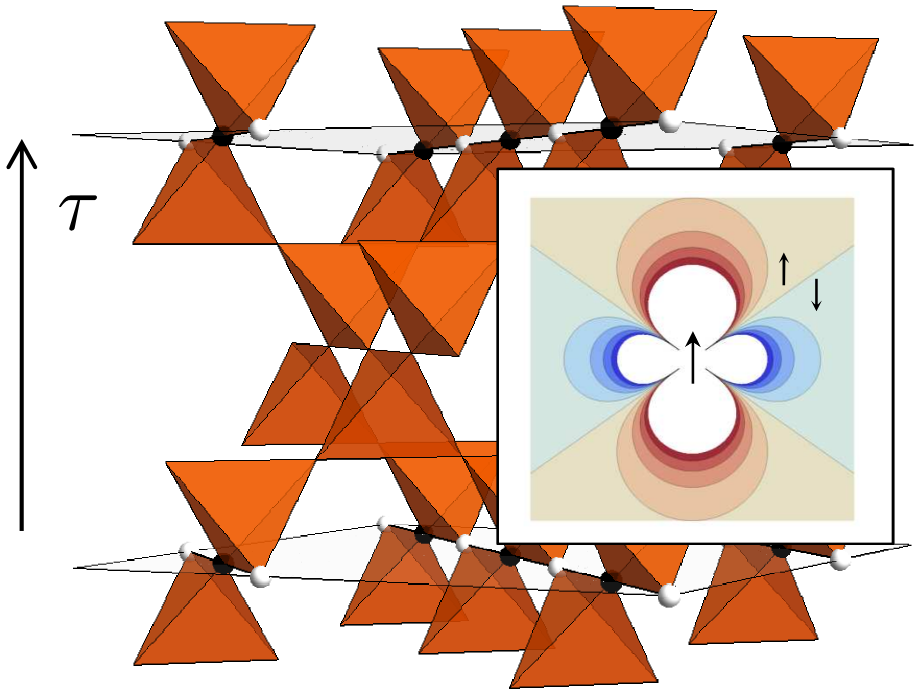
Continuum theory: Coulomb phase

Coulomb/superfluid phase: $\langle \psi \rangle \neq 0$

$$\psi \sim \mathrm{e}^{\mathrm{i}\phi}$$

Effective phase-only action:

$$\mathcal{L} = |\partial_\tau \phi|^2 + c^2(|\partial_x \phi|^2 + |\partial_y \phi|^2) + \cdots$$

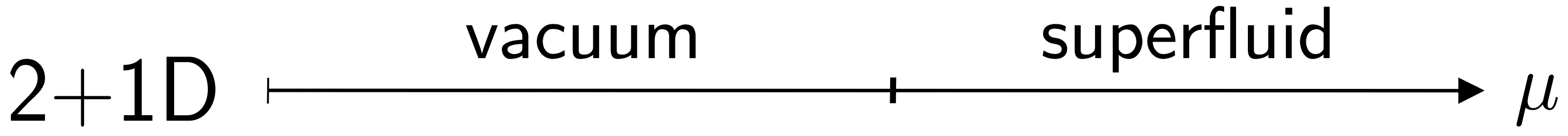


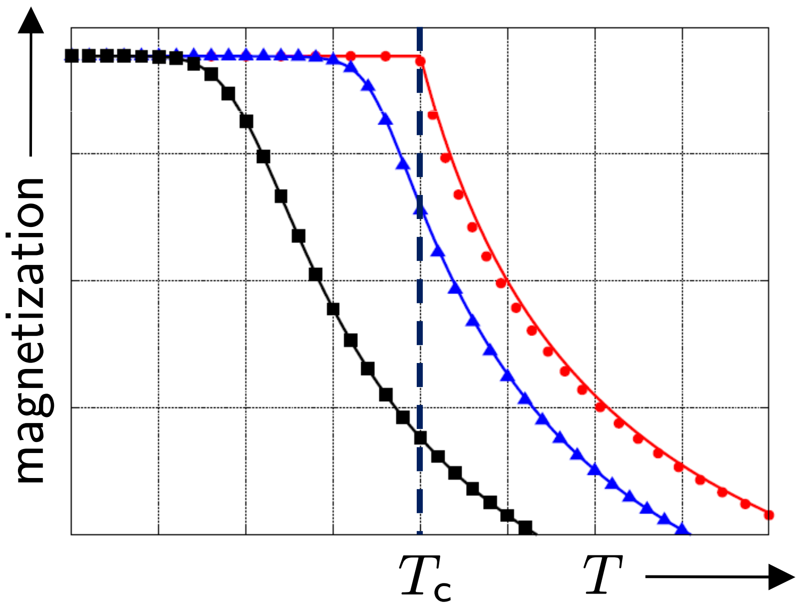
$$n_{\circ,\bullet} \sim (\partial_\tau \pm v\partial_y)\phi$$

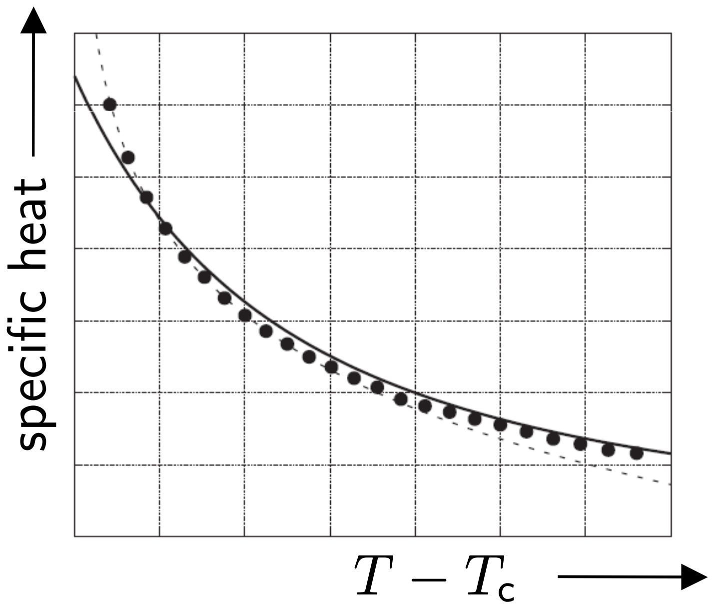
$$\langle n_\circ n_\circ \rangle \sim \frac{(\omega + v k_y)^2}{\omega^2 + c^2(k_x^2 + k_y^2)}$$

Critical theory

$$\mathcal{L}_{\text{critical}} = \psi^* \partial_\tau \psi + |\partial\psi|^2 - \mu|\psi|^2 + u|\psi|^4 + \cdots$$







Fisher and Hohenberg, Phys. Rev. B **37**, 4936 (1988)
Jaubert et al., Phys. Rev. Lett. **100**, 067207 (2008)

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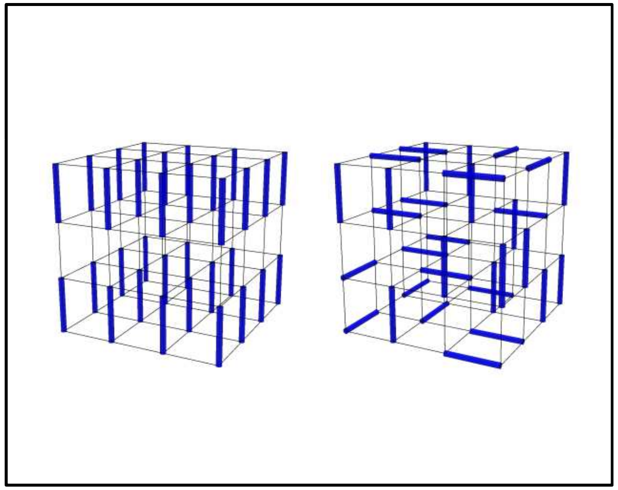
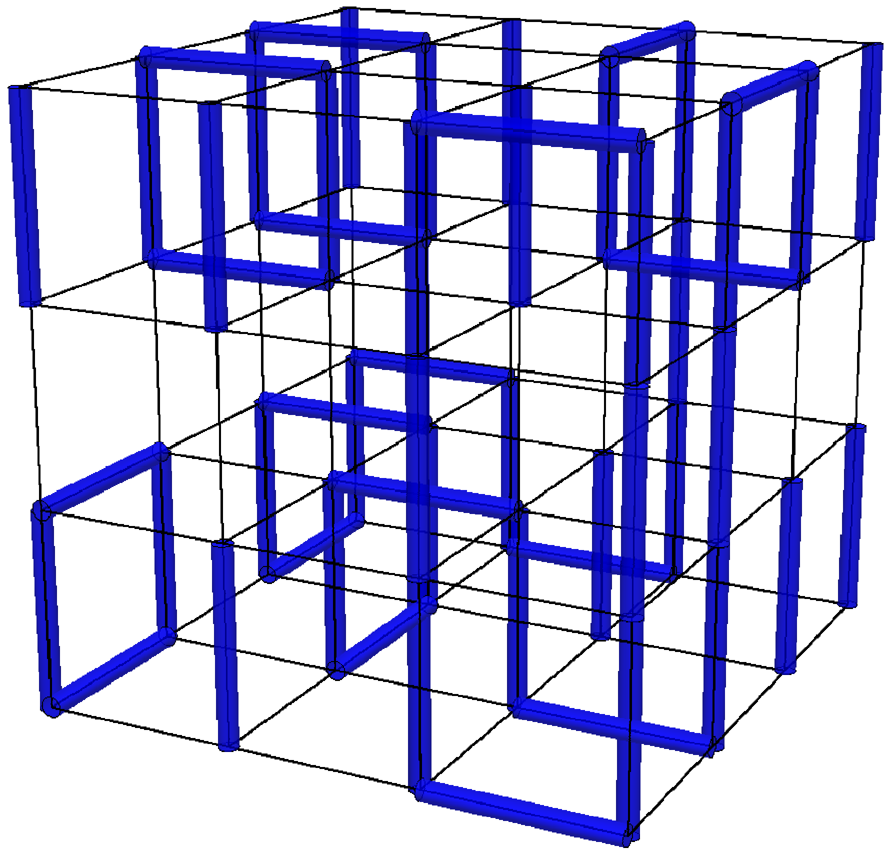


Classical dimer model

$$\boxed{\mathcal{E} = -UN_{\parallel}}$$

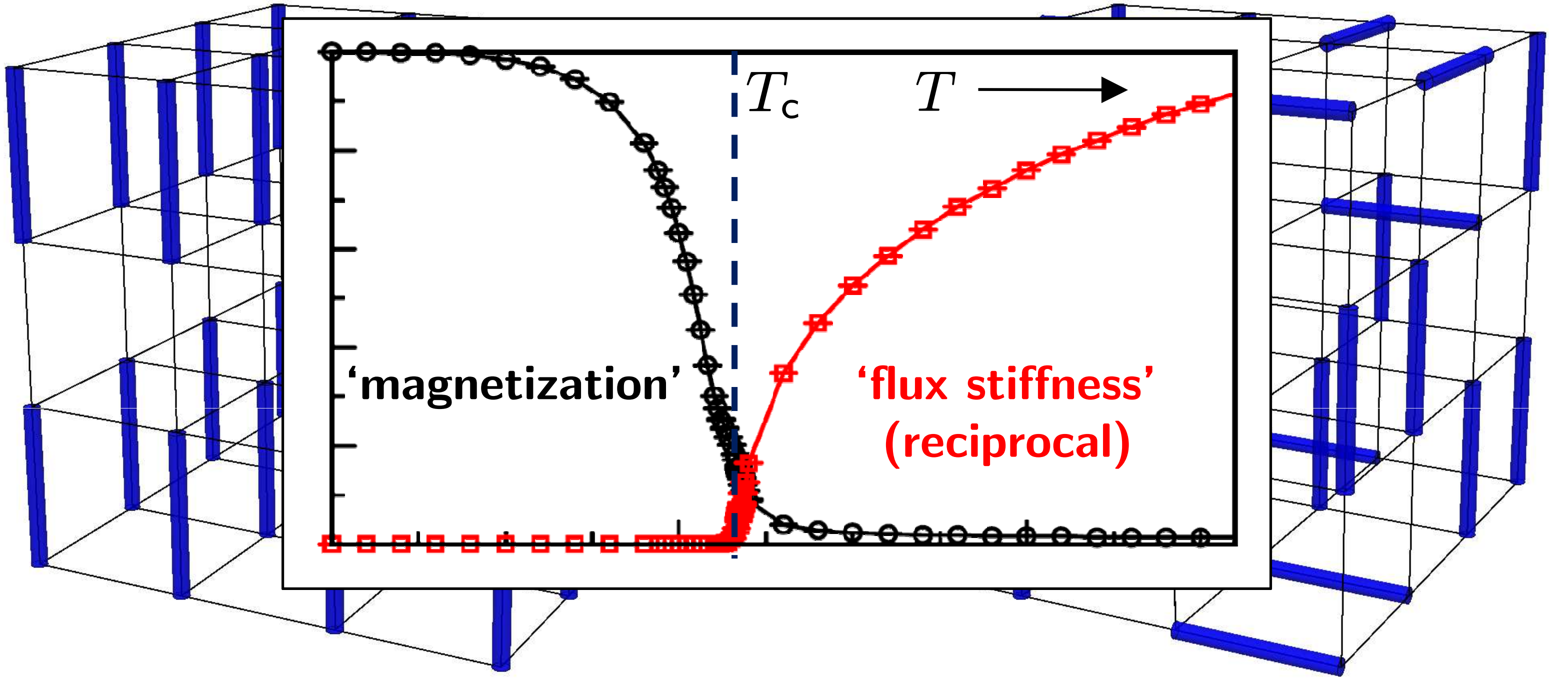
$U/T \to 0$: Coulomb

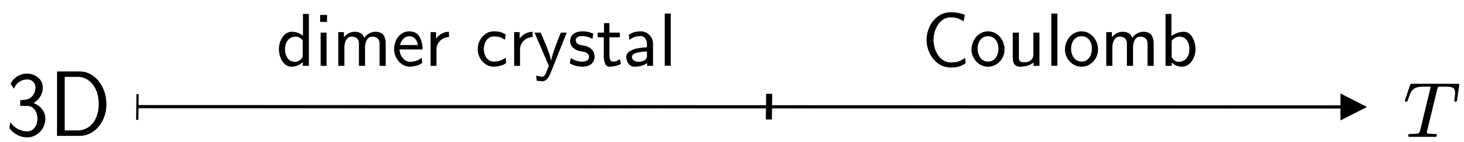
$U/T \to \infty$: order



Huse et al., Phys. Rev. Lett. **91**, 167004 (2003)

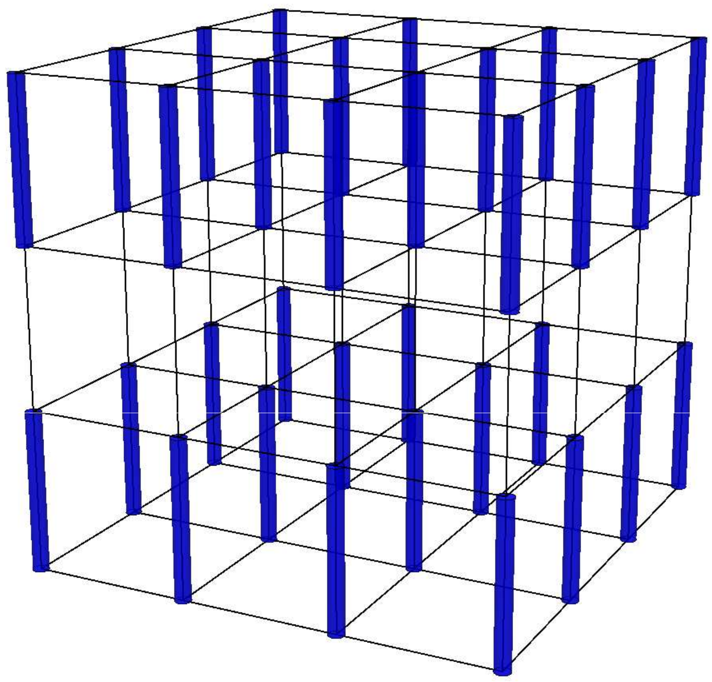
Coulomb–order transition

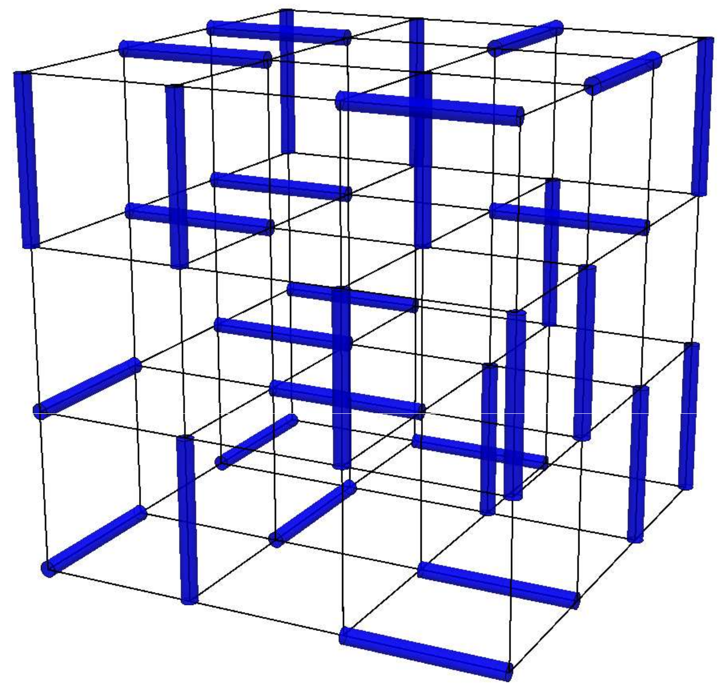




Alet et al., Phys. Rev. Lett. **97**, 030403 (2006)

Coulomb–order transition





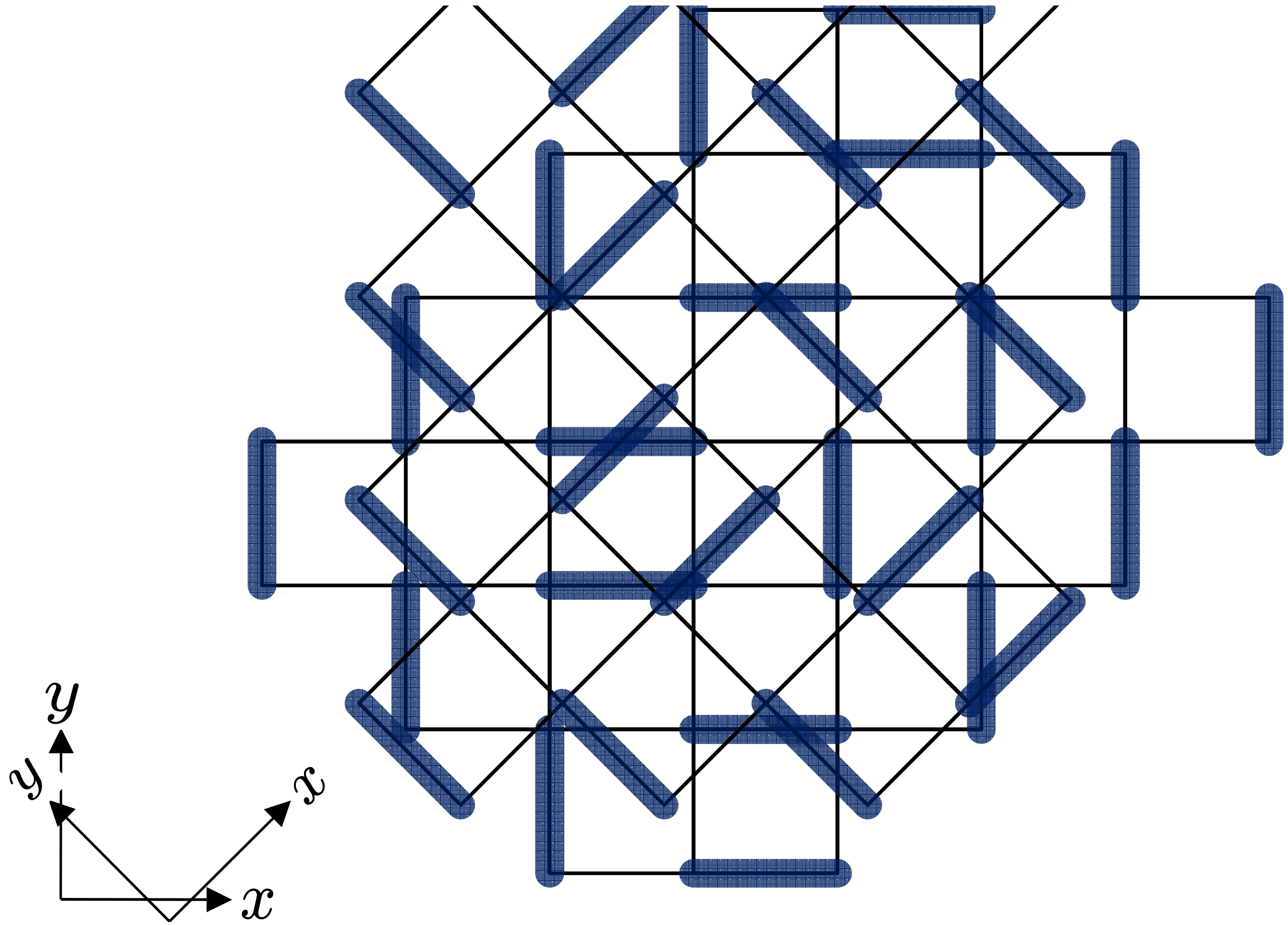
dimer crystal / Mott insulator | Coulomb superfluid

3D ⟶ T

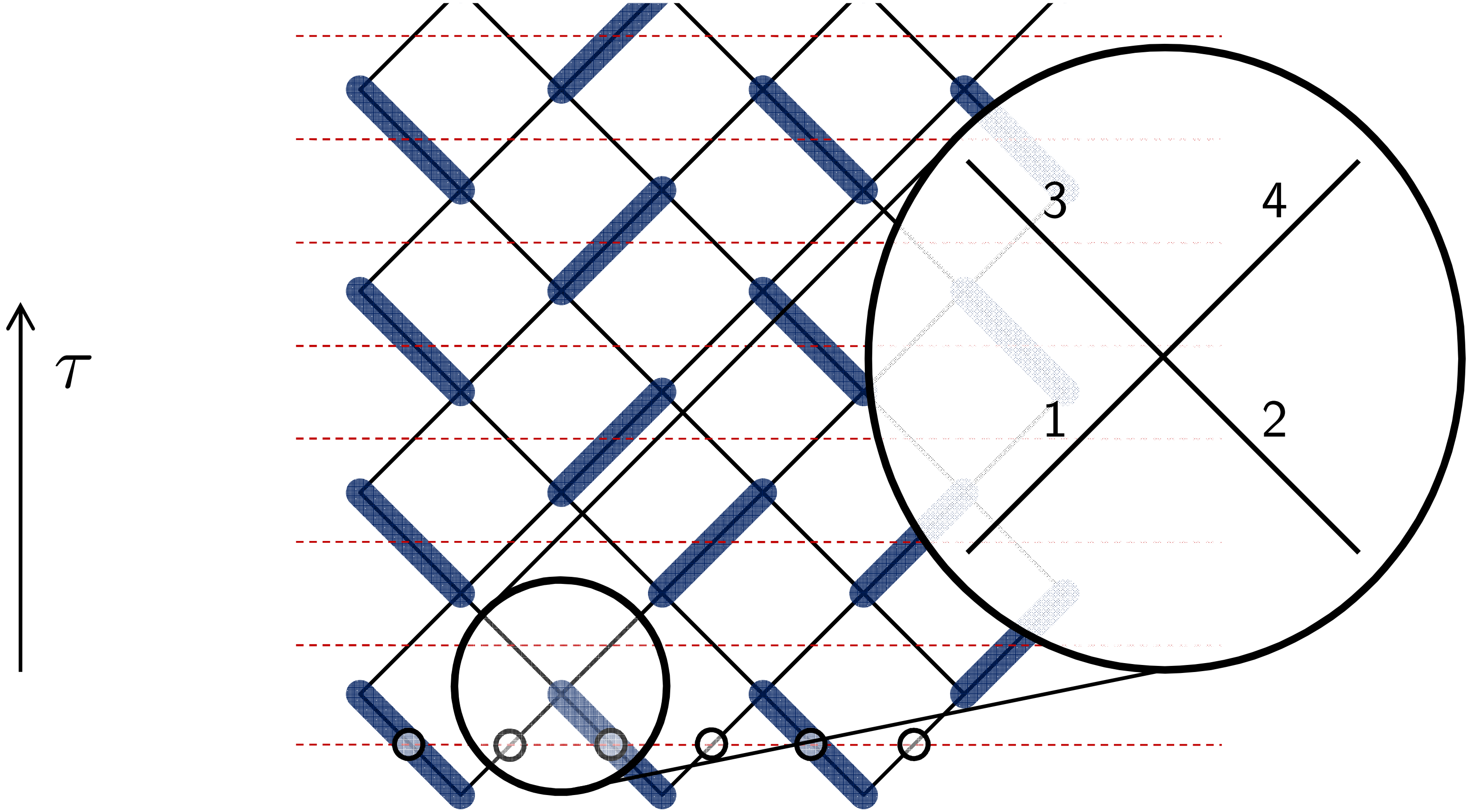
2+1D ⟶ t

SP and J. T. Chalker, PRL **101**, 155702 (2008)

Square-lattice dimers



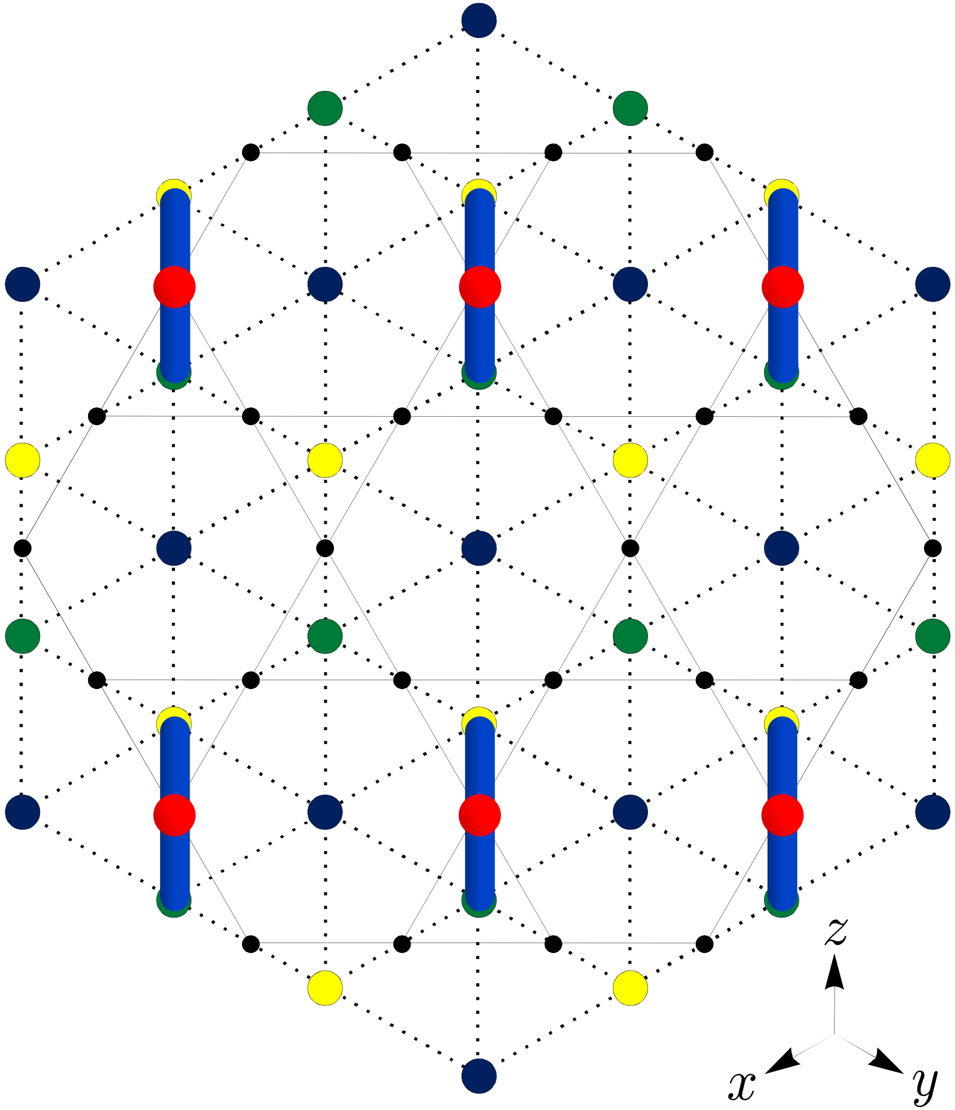
Square-lattice dimers

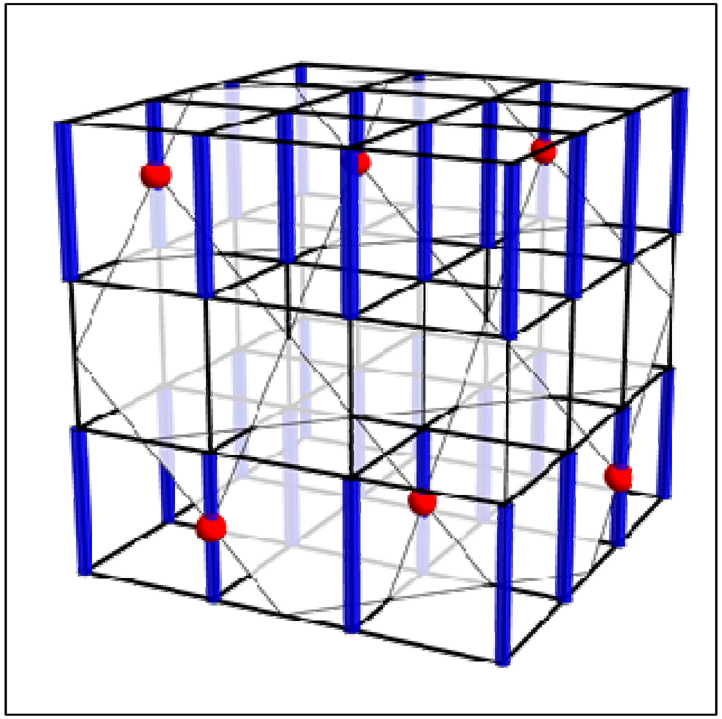


$$\sum_{\text{vertices}} n_{\text{afte}} \sum_{\text{vertices}} n_{\text{after}} = \sum_{\text{vertices}} (1 - n_{\text{before}}) \sum_{\text{rtices}} n_{\text{before}} \quad n_4 = 1$$

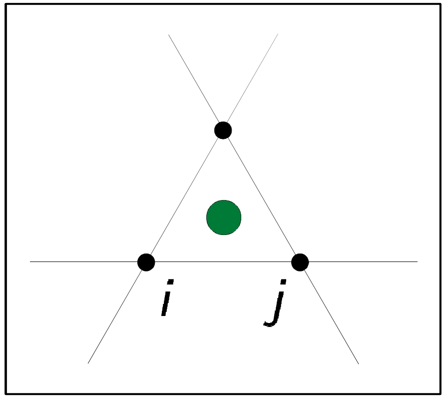
n_{before}

Cubic dimers to kagome bosons





Quantum Hamiltonian



$$\mathcal{H} = -\mu \sum_i n_i + \underbrace{\frac{U}{2}\sum_i n_i(n_i - 1) + U \sum_{\langle ij \rangle} n_i n_j}_{\text{Constraint: } U \to \infty}$$

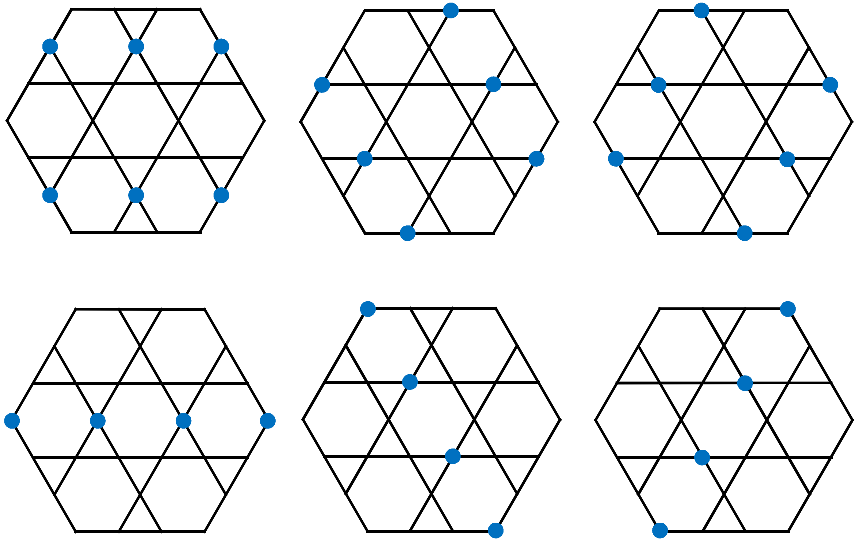
$$+ \sum_{ij} V_{ij} n_i n_j - \sum_{ij} t_{ij} b_i^\dagger b_j + \sum_{ijl} w_{ijl} n_i b_j^\dagger b_l + \cdots$$

Mott insulator — superfluid $\longrightarrow \frac{t}{V}$

(dimer crystal) — (Coulomb)

SP and J. T. Chalker, PRL 101, 155702 (2008)

Critical theory



$$\langle \psi \rangle \neq 0$$

Mott insulator — superfluid $\rightarrow t$

$\langle \boldsymbol{\varphi} \rangle \neq \mathbf{0}$ — $\langle \boldsymbol{\varphi} \rangle = \mathbf{0}$

dual (vector) vortex field $\boldsymbol{\varphi}$

SP and J. T. Chalker, PRL **101**, 155702 (2008)

Critical theory

$$\mathcal{L} = |(\boldsymbol{\nabla} - \mathrm{i}\mathbf{A})\boldsymbol{\varphi}|^2 + s|\boldsymbol{\varphi}|^2 + u(|\boldsymbol{\varphi}|^2)^2 + \Box|\boldsymbol{\nabla} \times \mathbf{A}|^2$$

- Noncompact U(1) gauge theory
- Emergent SU(2) symmetry

Mott insulator | superfluid $\longrightarrow t$

$\langle \boldsymbol{\varphi} \rangle \neq \mathbf{0}$ | $\langle \boldsymbol{\varphi} \rangle = \mathbf{0}$

dual (vector) vortex field $\boldsymbol{\varphi}$

SP and J. T. Chalker, PRL **101**, 155702 (2008)

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Cubic lattice dimer model, Phys. Rev. Lett. **101**, 155702 (2008), arXiv:0907.1564 (to appear in PRB)