Topological Phases in One Dimension

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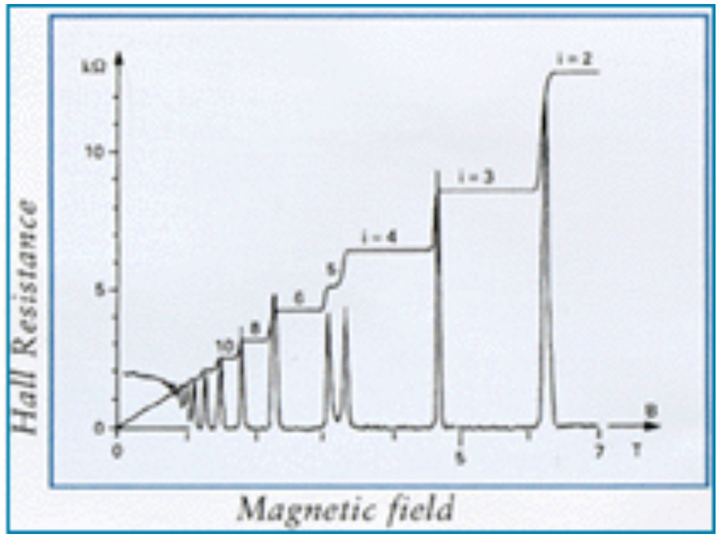
arXiv:1008.4138

Topological phases in 2 dimensions:

- *Integer quantum Hall effect*
    - quantized σ_{xy}
    - robust chiral edge modes



- *Fractional quantum Hall effect*
    - fractionally charged quasi-particles
    - robust chiral edge modes

- Quantum spin Hall, topological insulators, etc.
    - free electrons, spin-orbit coupling

The Majorana Wire

- spin-less p-wave superconductor
- tight-binding model:

j = 1 2 3 4 5 6

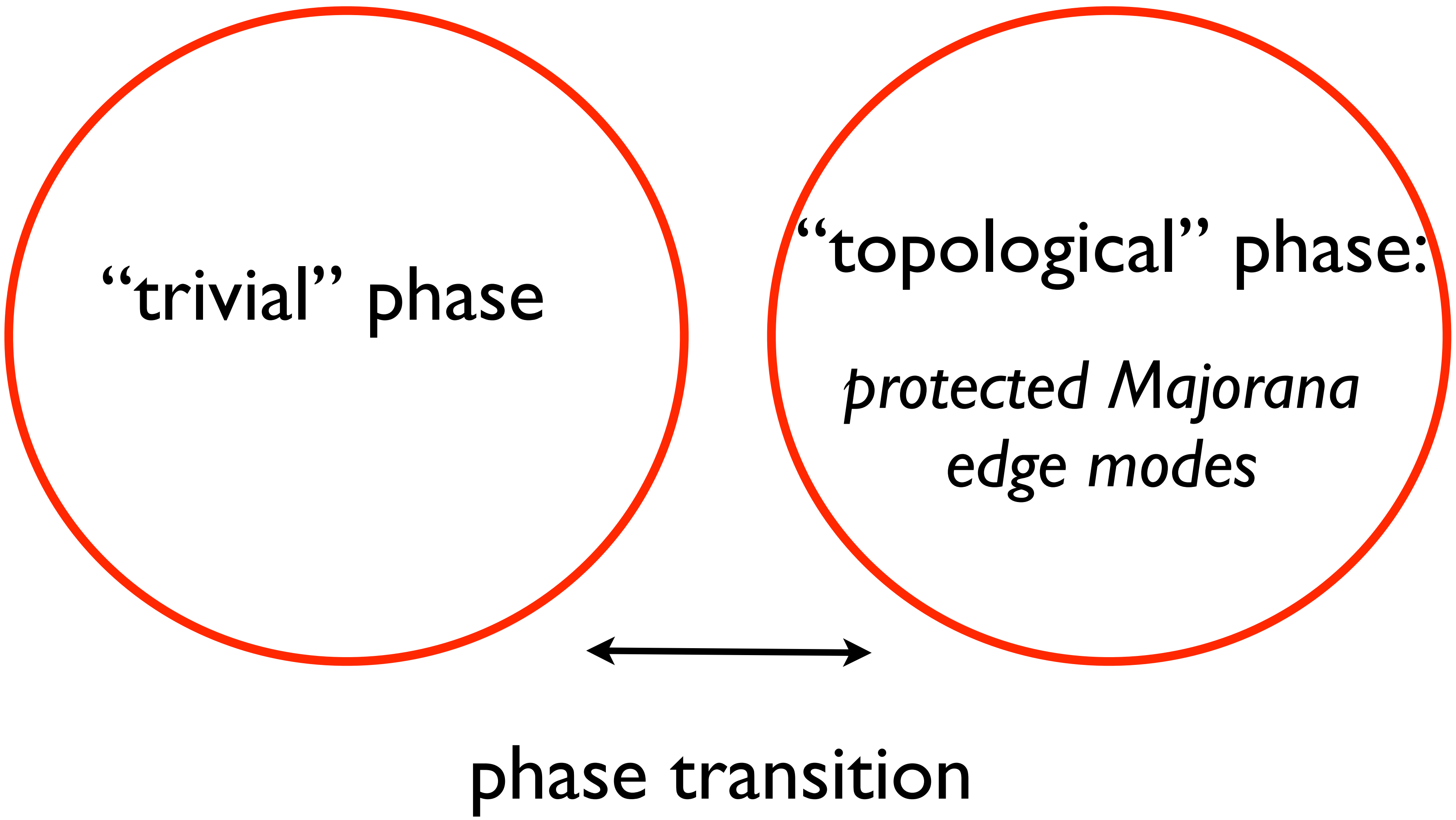
$$H = \sum_j \Big(\mu \, a_j^\dagger a_j + t(a_j a_{j+1}^\dagger + a_{j+1} a_j^\dagger) + \Delta(a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger) \Big)$$

chemical potential

hopping

pairing

Gapped Hamiltonians:



Re-write in terms of Majorana modes:

$$\langle a_j, a_j^\dagger \rangle \longrightarrow \langle c_{2j-1}, c_{2j} \rangle$$

$$c_{2j-1} = -i(a_j + a_j^\dagger)$$
$$c_{2j} = a_j + a_j^\dagger$$

$$\{a_1, \ldots, a_N, a_1^\dagger, \ldots, a_N^\dagger\} \longrightarrow \{c_1, \ldots, c_{2N}\}$$

N fermion creation / annihilation operators

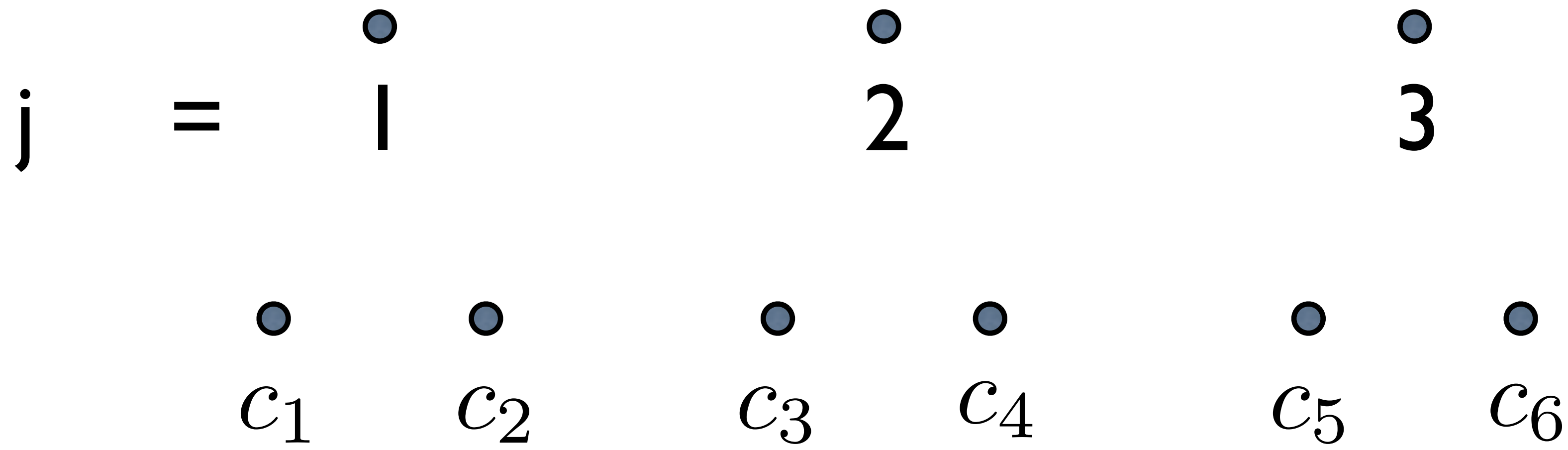
2N Hermitian Majorana operators

- our Hamiltonian can then be written as a quadratic form in the Majoranas:

$$H = \sum_{m,n=1}^{2N} A_{mn}\, c_m c_n$$

- graphical representation:

Hilbert Space:

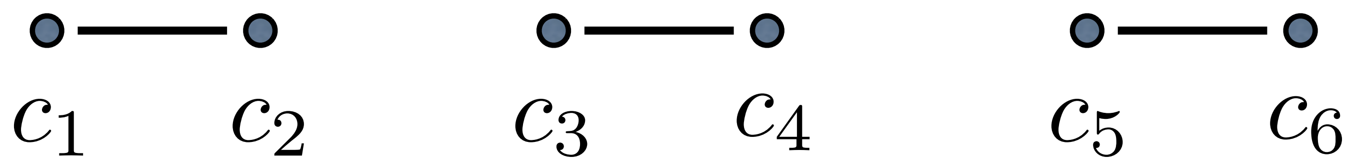


Hamiltonian:

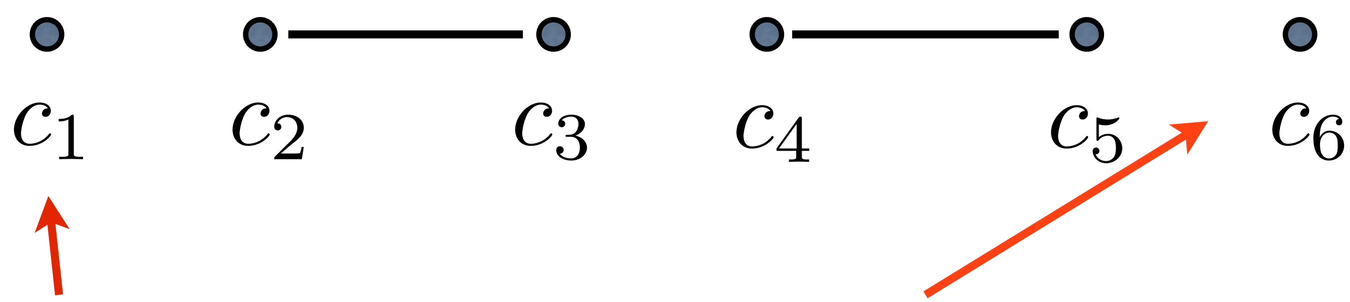
$$= \frac{i}{2}\left(c_1 c_2 + c_3 c_4 + c_5 c_6\right)$$

$$= a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \longleftarrow \text{decoupled}$$

trivial phase:



topological phase:



dangling Majorana modes

$$a = \frac{1}{2}(c_1 + ic_6) \qquad \longrightarrow \qquad \text{double ground state degeneracy}$$

$$a^\dagger = \frac{1}{2}(c_1 - ic_6)$$

Recap:

Itinerant spin-less fermions in one dimension have two phases. The non-trivial "topological" one is characterized by having Majorana edge modes at its endpoints.

What is left:

1) interactions?

2) symmetries?

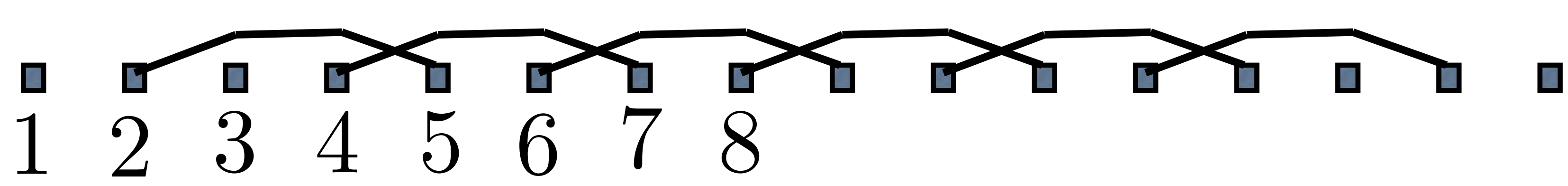
either generic (like time reversal or particle-hole) or some arbitrary symmetry group G (like SU(2) in spin chains)

Majorana chain with time reversal symmetry

- spin-less fermions as before, with

$$T: \begin{array}{c} a_j \to a_j \\ a_j^\dagger \to a_j^\dagger \end{array} \qquad \longleftrightarrow \qquad T: c_k \to (-1)^k c_k$$

- non-interacting (i.e. quadratic fermion) analysis gives infinitely many phases, characterized by an integer n$\in \mathbb{Z}$



we showed that with interactions, $\mathbb{Z}$ is broken down to $\mathbb{Z}_8$

$$n \to n \mod 8$$

showed this by turning on quartic interactions and finding an explicit path in interacting Hamiltonian space which connects n and n+8.

But how to handle interactions in general?

Matrix Product States:

- Bosonic spin chains with local local spins s

$$\Psi(s_1, \dots, s_N) = \mathrm{Tr}(A_{s_1} \dots A_{s_N})$$

$$s_i \in \{-s/2, -s/2+1, \dots, s/2\}$$

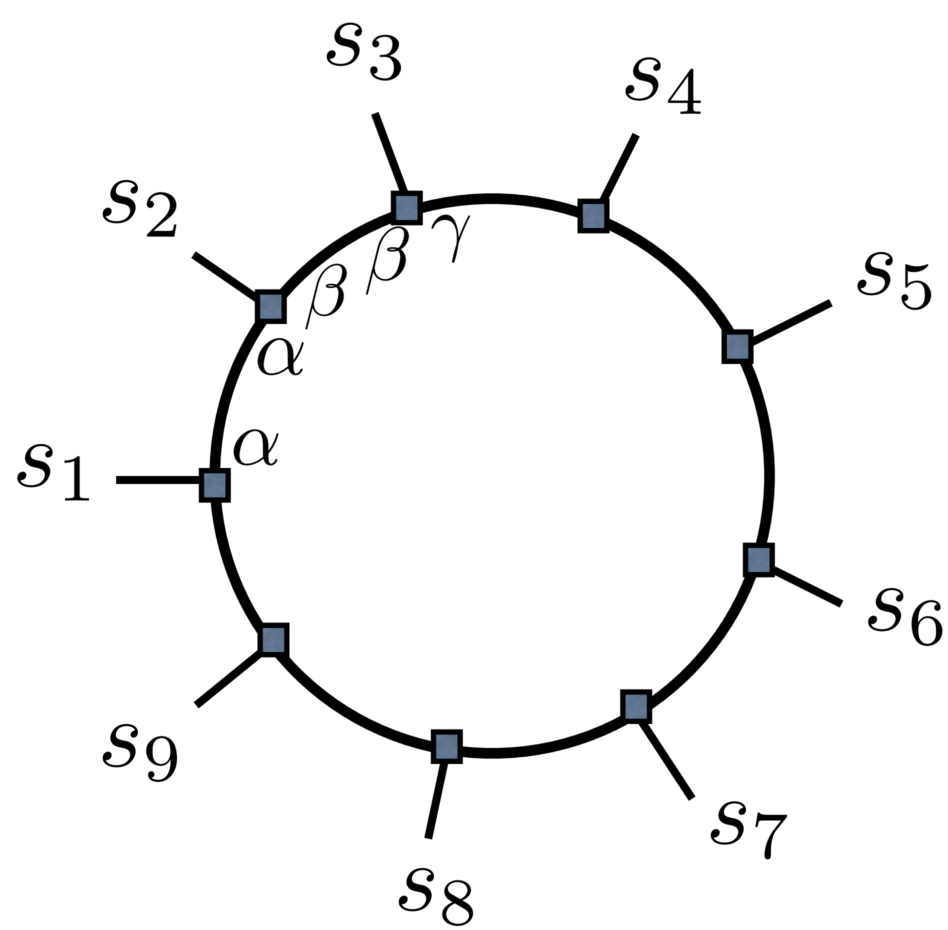
$$A_{s_i} : D \times D \text{ matrices}$$

(can be generalized to fermionic systems via Jordan-Wigner transform)

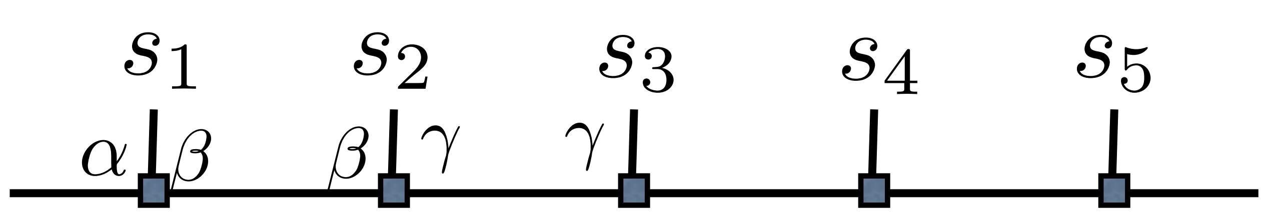
- Ground states of gapped Hamiltonians can be approximated arbitrarily well by MPS with small D (Hastings 2006)

Tensor network contraction picture:

Closed MPS:



Open MPS:



$$\alpha, \beta, \gamma \in \{1, \ldots, D\}$$

Virtual degrees of freedom <=> edge modes

Theorem: If G is a symmetry of the original states, then G has a *projective* action on the virtual degrees of freedom.

(*projective* means that group relations are obeyed only up to phases, which one might not be able to gauge away)

Thus, if G is a symmetry of the Hamiltonian, then there is a projective action of G on the edge modes.

Generally, there is a discrete set of classes of projective representations of G, and these classes correspond to different phases. In fact, they enumerate *all* gapped phases of Hamiltonians with that symmetry group.

Example: AKLT Hamiltonian for spin-1 Heisenberg model

G = SO(3), but edge states in the non-trivial (Haldane) phase are half-integer spins. These are representations of SU(2), and only projective representations of SO(3).

Example: Majorana chain with time reversal symmetry

G = <1,T, P, PT> where $P = (-1)^F$

If n mod 8 is the previously discussed topological index then:

- n mod 2: P bosonic vs. fermionic
- n mod 4: T commuting/anti-commuting with P
- n mod 8: $T^2 = +1$ or $T^2 = -1$