Fractional topological insulators

Claudio Chamon

Christopher Mudry - PSI

Titus Neupert - PSI

Shinsei Ryu - Berkeley

Luiz Santos - Harvard

PRL 106, 236804 (2011)

PRB 84, 165107 (2011)

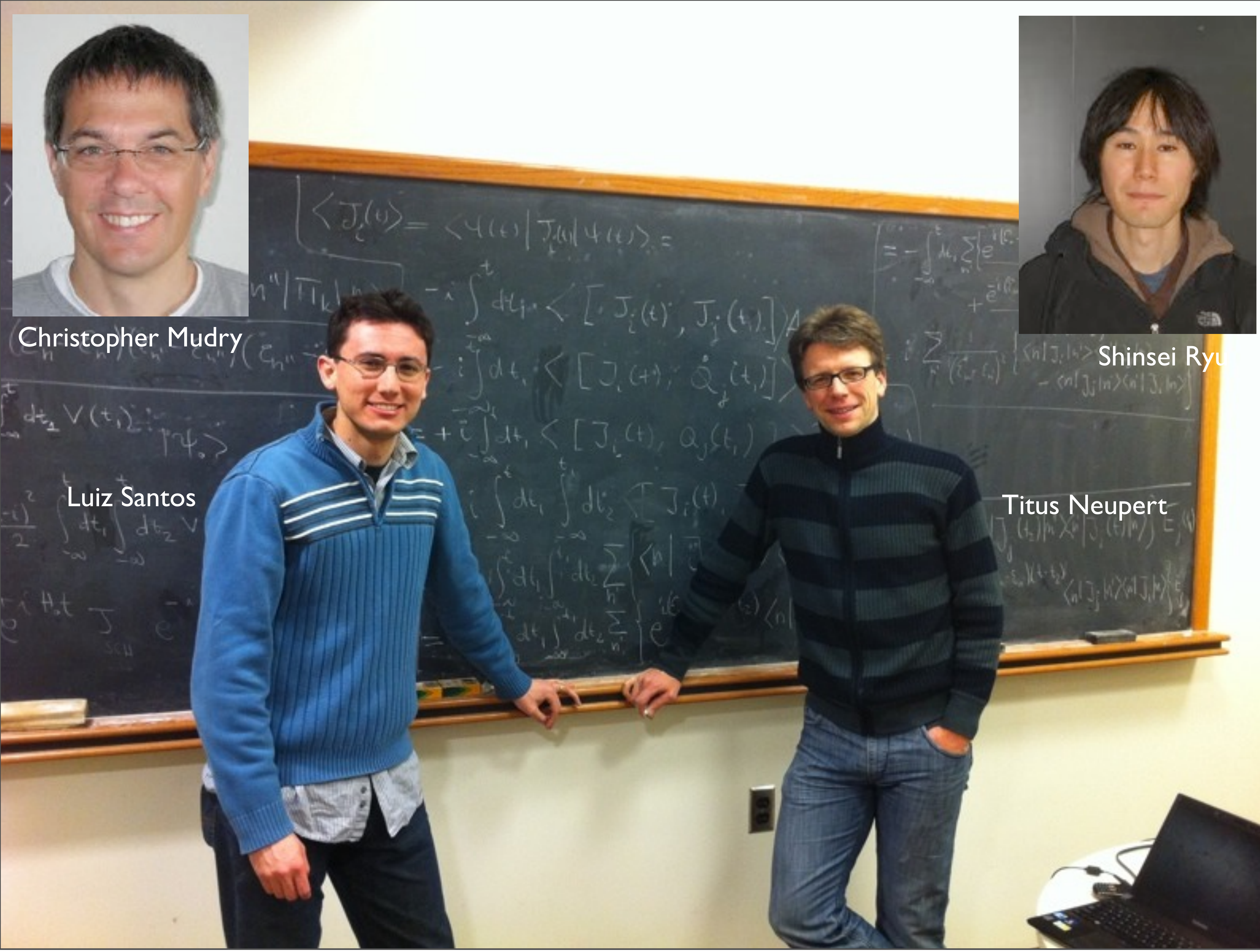
PRB 84, 165138 (2011)

PRL 108, 046806 (2012)

arXiv:1202.5188

support: DOE

BOSTON UNIVERSITY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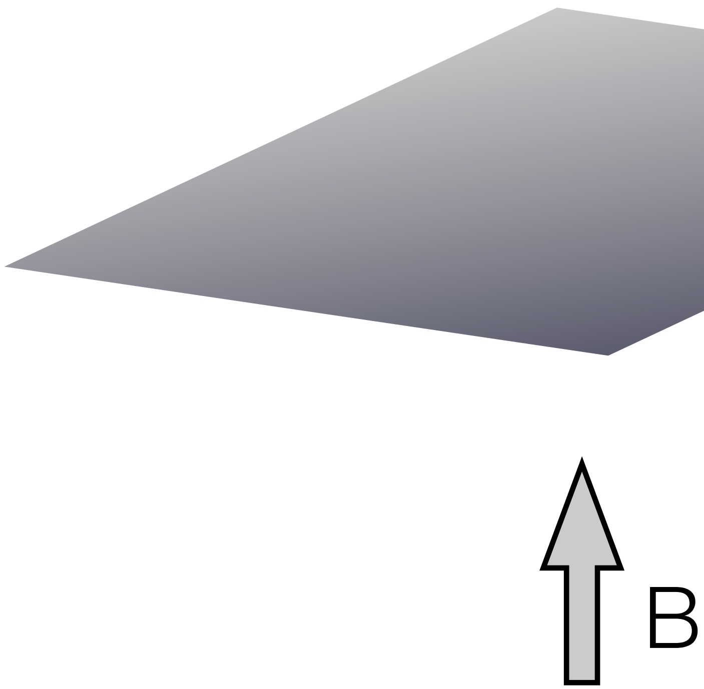
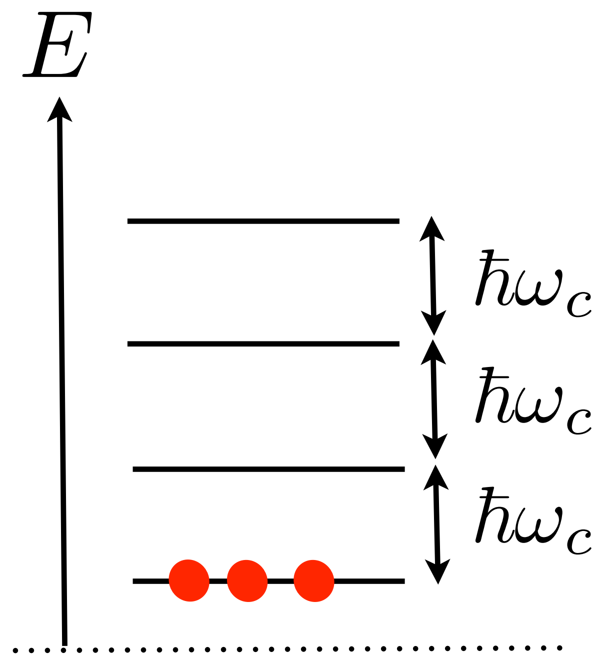
Landau flat bands

Landau levels





degeneracy $N_\phi = B\,\mathcal{A}/\phi_0$

Landau flat bands are interesting!

Partially filled Landau levels ⇨ fractional quantum Hall effect (FQHE)

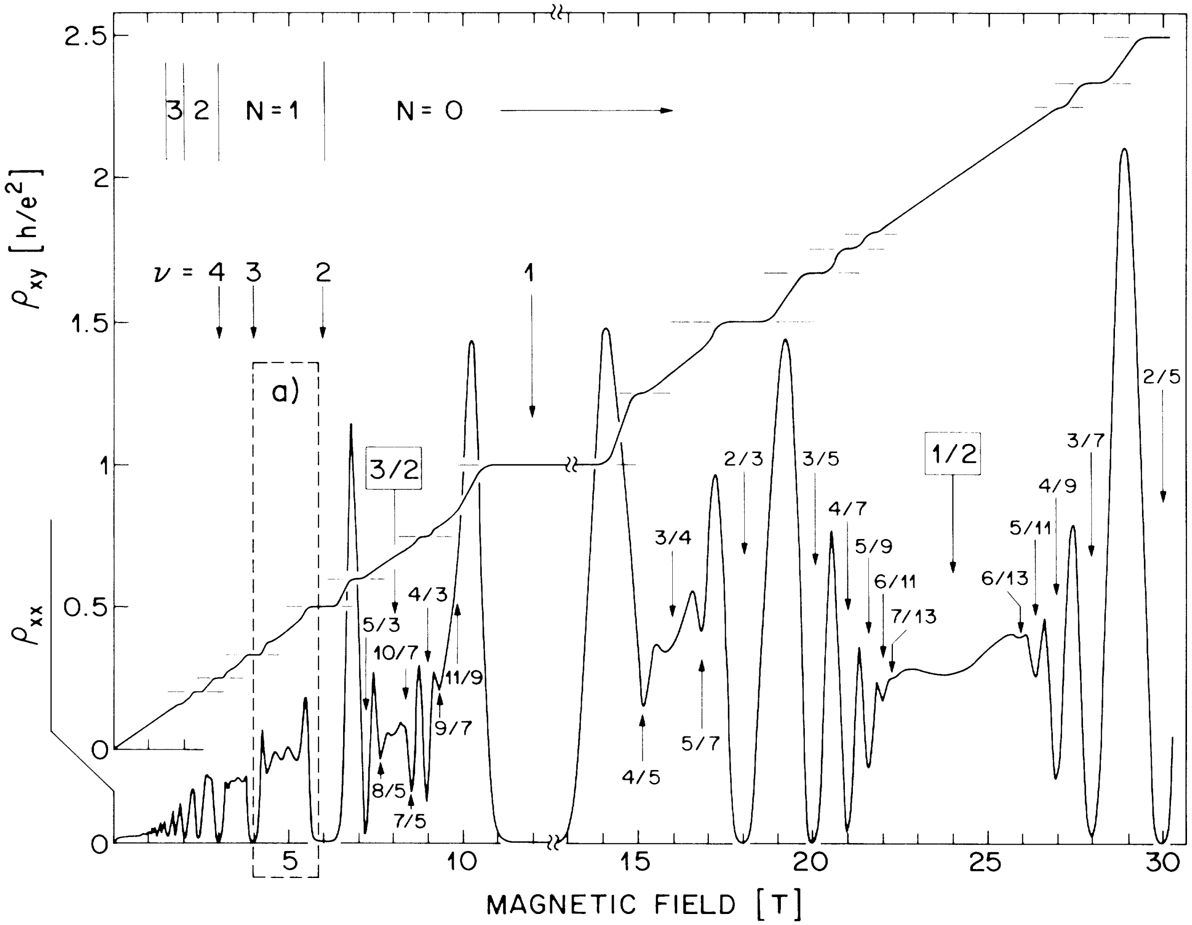


FIG. 1. Overview of diagonal resistivity ρ_{xx} and Hall resistance ρ_{xy} of sample described in text. The use of a hybrid magnet with fixed base field required composition of this figure from four different traces (breaks at $\simeq$12 T). Temperatures were $\approx$150 mK except for the high-field Hall trace at T=85 mK. The high-field ρ_{xx} trace is reduced in amplitude by a factor 2.5 for clarity. Filling factor ν and Landau levels N are indicated.

source:
Willett et al, PRL 1987

Are there other ways to get flat bands?

Are they as interesting as Landau levels?

Early flat bands

D. Weire and M. F. Thorpe, PRB 4, 2508 (1971)
J. P. Straley, PRB 6, 4086 (1972)

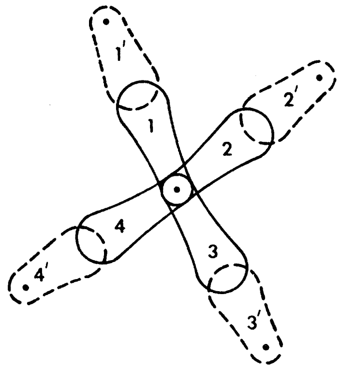


FIG. 3. Coefficients of the basis functions 1–4 associated with a given atom, in the expansion of a given wave function, are the elements of the vector $\vec{u}$ defined in the text. The coefficients of 1′–4′ form the vector $\vec{v}$.

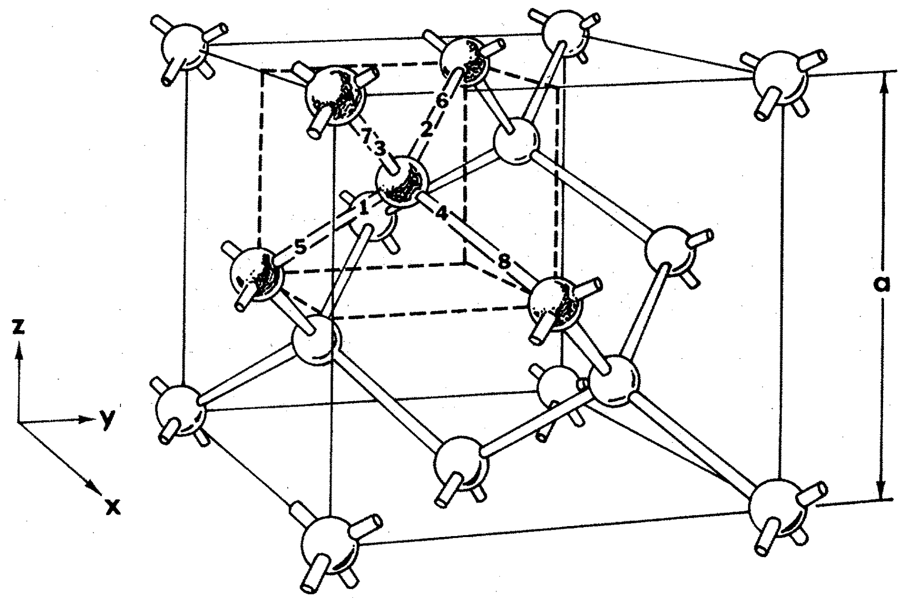


FIG. 13. A unit cell of the diamond cubic structure, illustrating the notation used to set up the secular determinant.

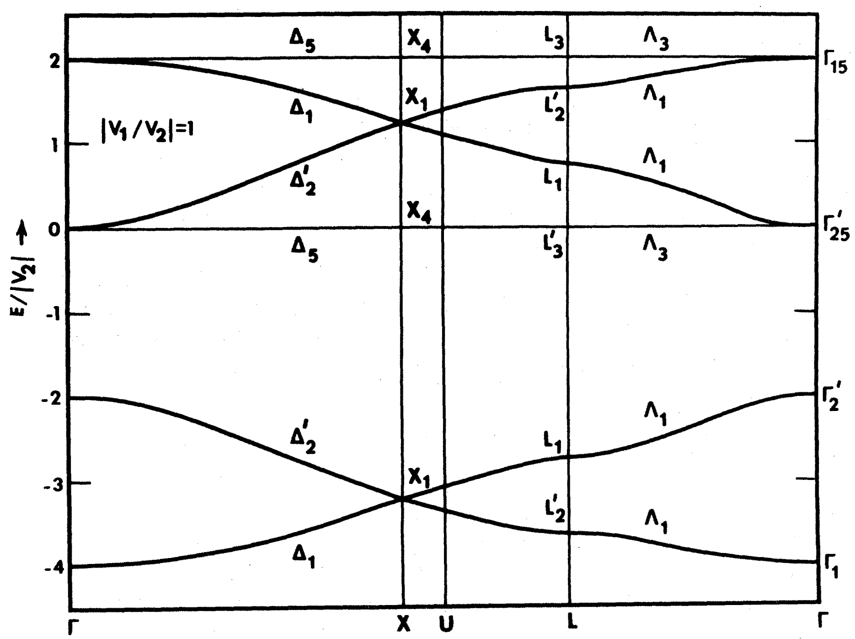


FIG. 11. Band structure for the special case of the diamond cubic structure, with $V_2=-1$, $V_1=-1$.

More early flat bands

E. Dagotto, E. Fradkin, and A. Moreo, Phys. Lett. B **172**, 383 (1986).

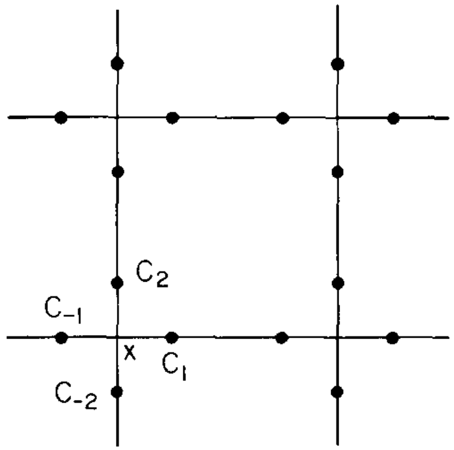
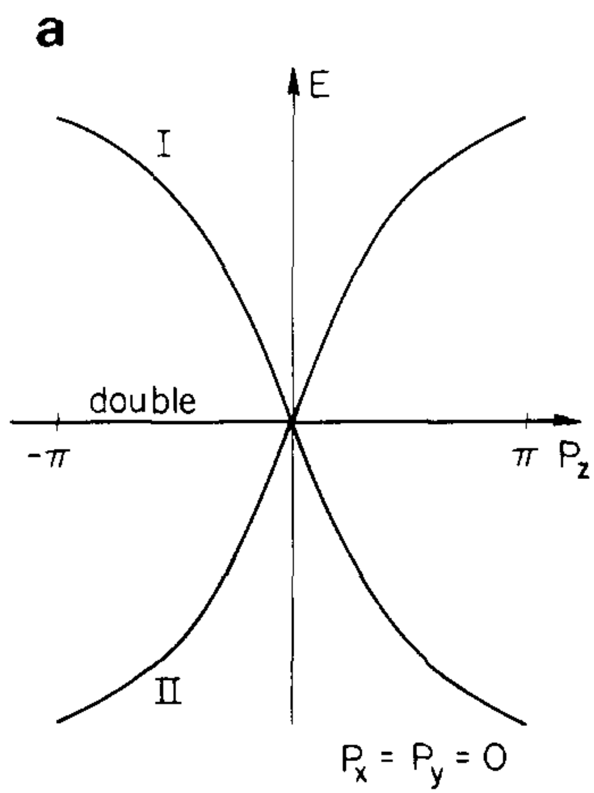


Fig. 2. A two-dimensional example showing the position of the variables used in eq. (1).

Of the generic form in
J. P. Straley, PRB 6, 4086 (1972)

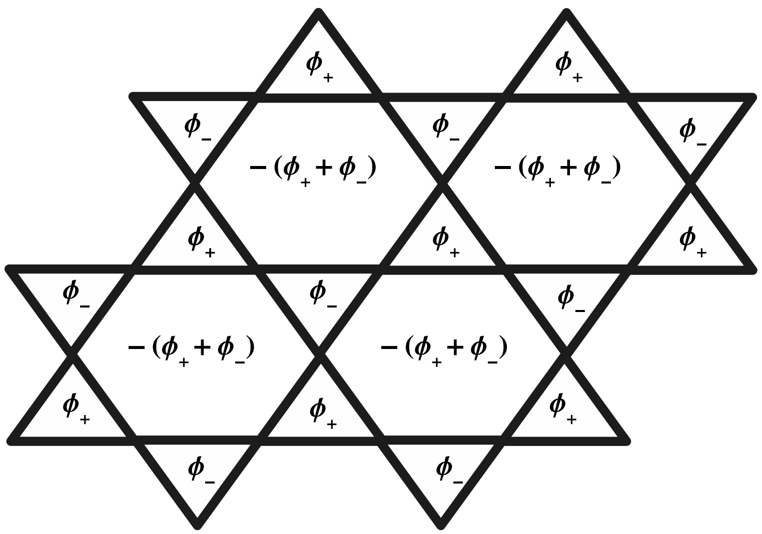


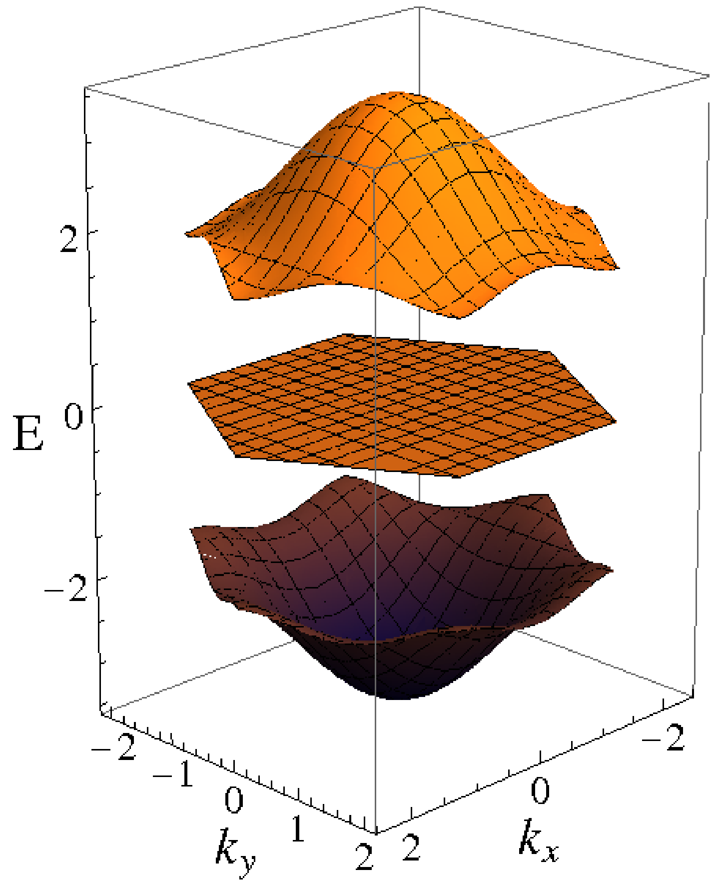
Motivation was to get Weyl fermions on the lattice, with a single cone

Possible flat band energies in the Kagome model

D. Green, L. Santos, and C. Chamon, PRB 2010

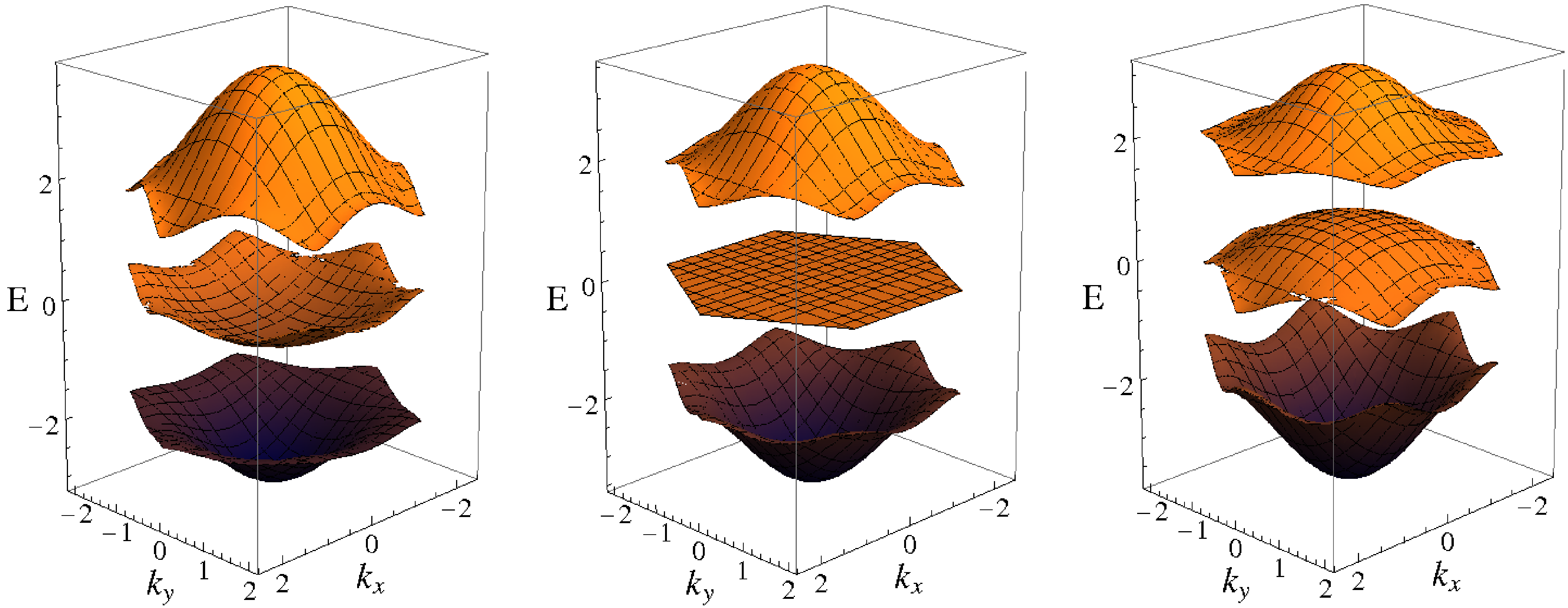
$E = -2g$	Flat very bottom band	$\phi_\pm = 2\pi\, n_\pm$
$E = +2g$	Flat very top band	$\phi_\pm = \pi\,(2n_\pm + 1)$
$E = 0$	Flat middle band	$\phi_+ + \phi_- = \pi \pmod{2\pi}$





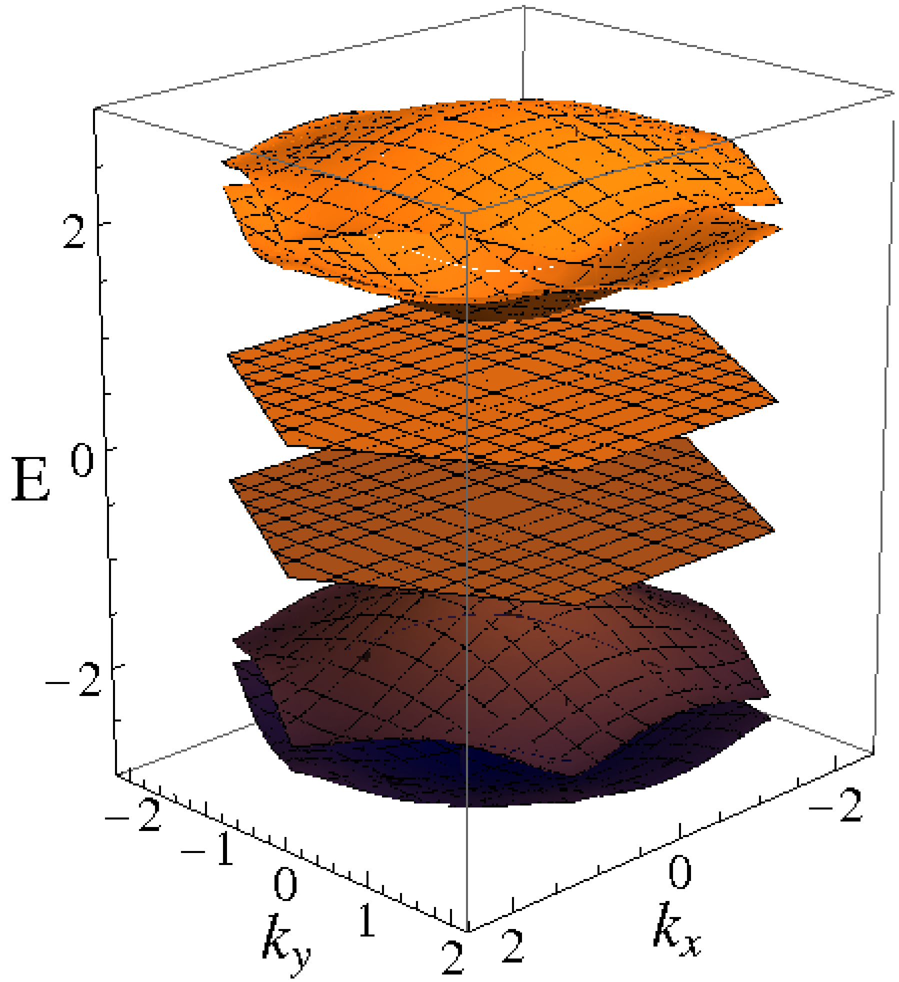
Flat band as a “critical point”

$$\phi_+ = \phi_- = 3\,(\pi/2 - \epsilon)$$



$\epsilon < 0$ $\epsilon = 0$ $\epsilon > 0$

Can go on and on...
Parallel flat bands- honeycomb lattice w/ 3 flavors



- Are there other ways to get flat bands?

- Are they as interesting as Landau levels?

Are there other ways to get flat bands?

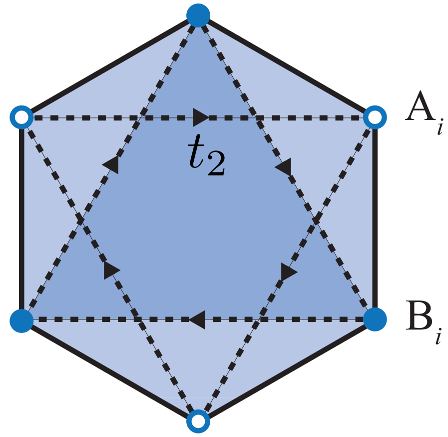
Are they as interesting as Landau levels?

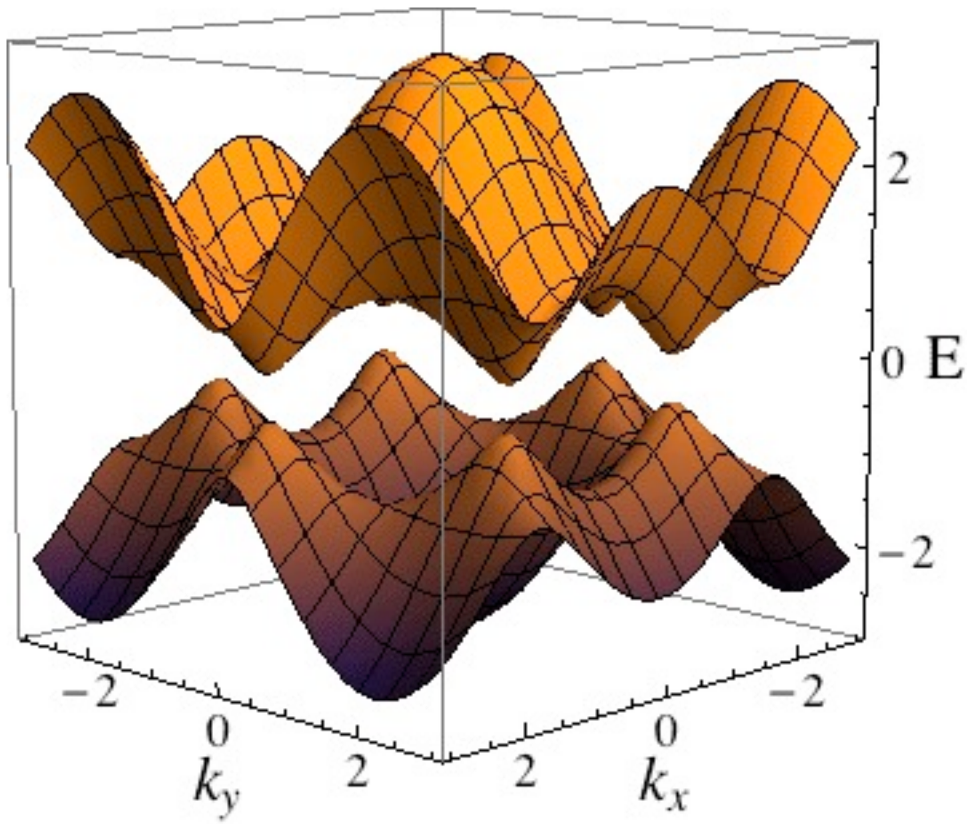
- ☐ Are they as interesting as Landau levels?

Up to that point, had examples of flat bands with <u>zero</u> Chern number

Quantum Hall effect w/o Landau levels

F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).





$$\sigma_{xy} = -1$$

$$\sigma_{xy} = +1$$

Can one flatten bands with a Chern number?

YES!

T. Neupert, L. Santos, C. Chamon, and C. Mudry, PRL 2011

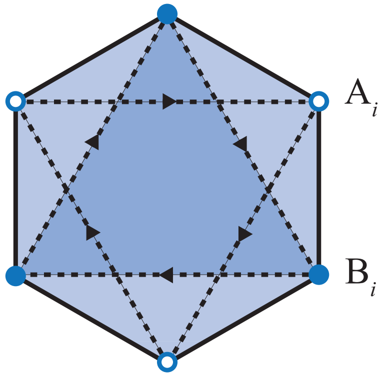
E. Tang, J. W. Mei, and X. G. Wen, PRL 2011

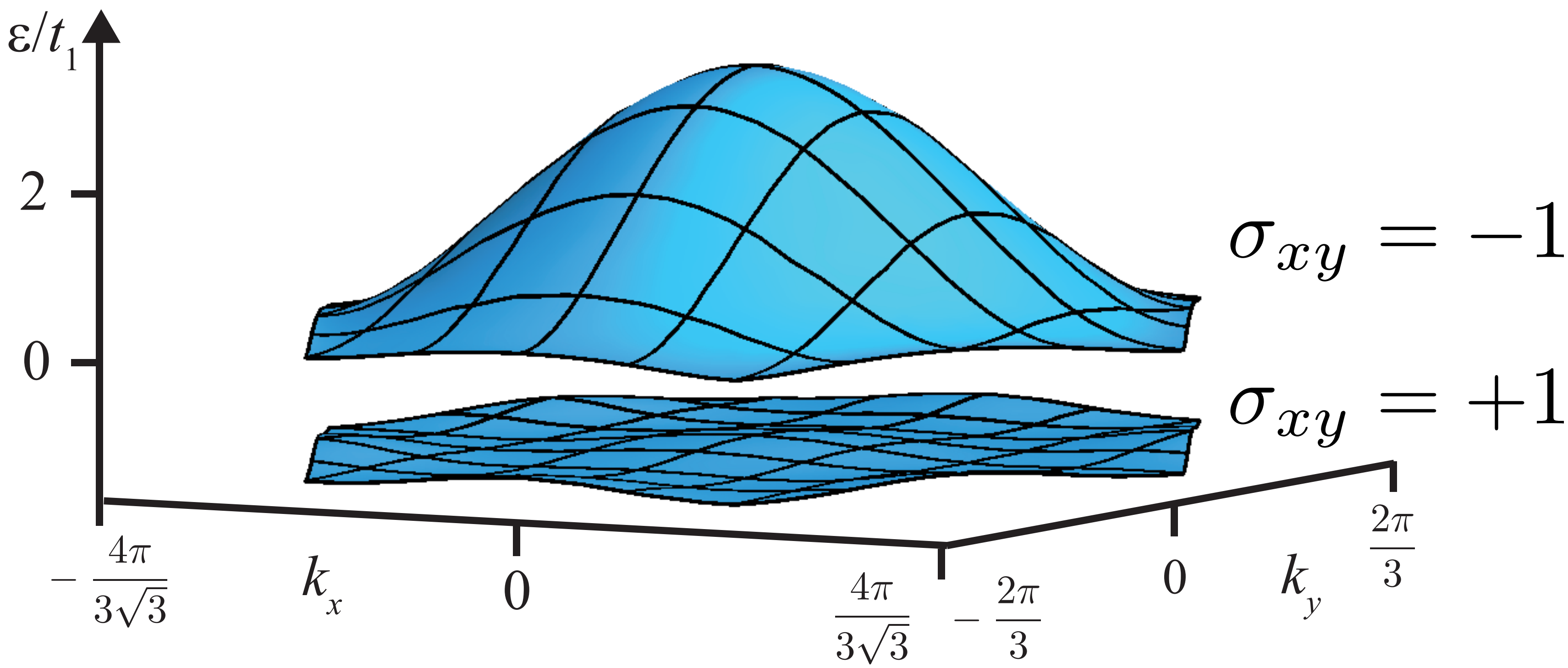
K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, PRL 2011

Quantum Hall effect w/o Landau levels

$$t_2/t_1 = \frac{\sqrt{43}}{12\sqrt{3}} \approx 0.315495 \qquad \cos\Phi = \frac{1}{4}\,\frac{t_1}{t_2}$$

$$\delta_-/\Delta = 1/7$$

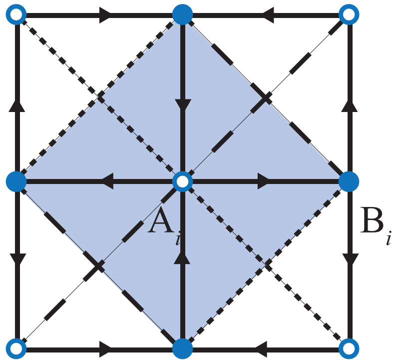


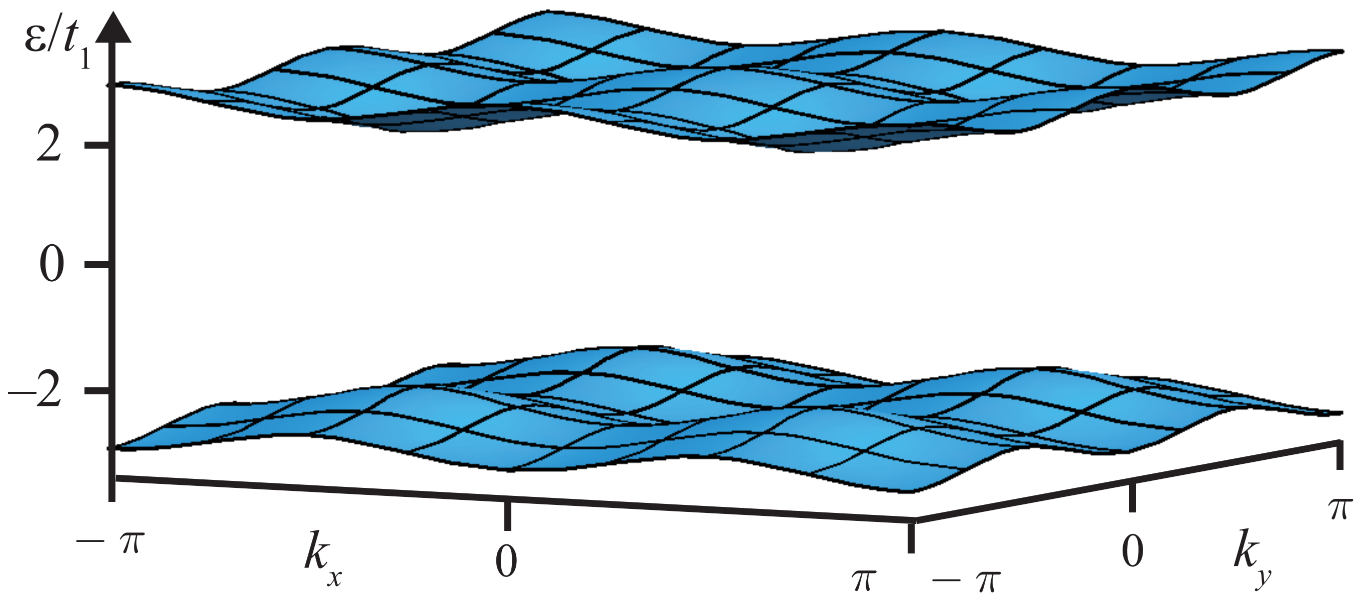


Quantum Hall effect w/o Landau levels

$$t_2/t_1 = \frac{1}{\sqrt{2}}$$

$$\delta_-/\Delta \approx 1/5$$

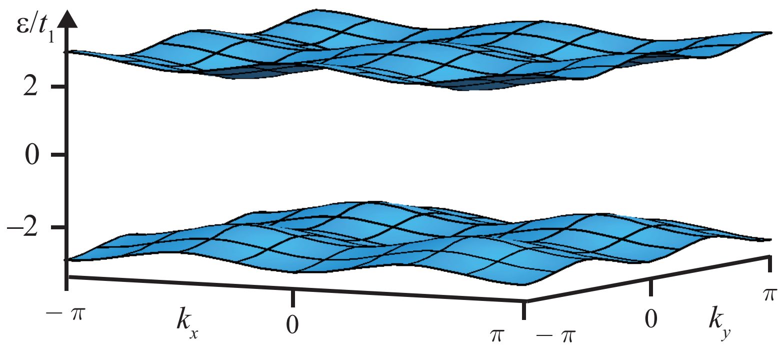


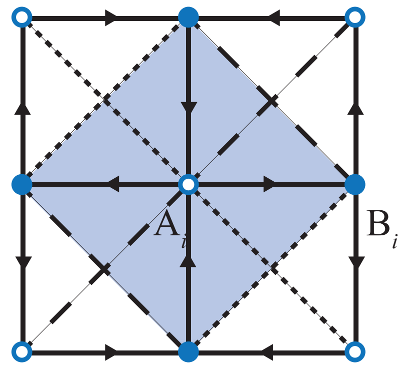


$$\sigma_{xy} = -1$$

$$\sigma_{xy} = +1$$

Can one get perfectly flat bands?





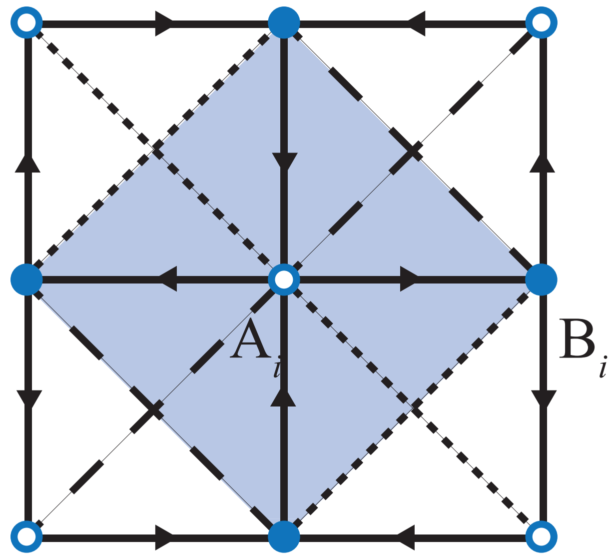
$$H_0 = \sum_{\boldsymbol{k}\in \mathrm{BZ}} \psi^\dagger_{\boldsymbol{k}} \mathcal{H}_{\boldsymbol{k}} \psi_{\boldsymbol{k}}, \qquad \mathcal{H}_{\boldsymbol{k}} = \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}$$

$$\psi^\dagger_{\boldsymbol{k}} = \left(c^\dagger_{\boldsymbol{k},\mathrm{A}}, c^\dagger_{\boldsymbol{k},\mathrm{B}} \right)$$

$$\mathcal{H}^{\mathrm{flat}}_{\boldsymbol{k}} := \frac{\mathcal{H}_{\boldsymbol{k}}}{|\varepsilon_{-,\boldsymbol{k}}|} \quad \Rightarrow \text{In real space, hoppings decay exponentially with distance}$$

Is there a FQHE when flat bands with non-zero Chern number are partially fixed?

Add interactions

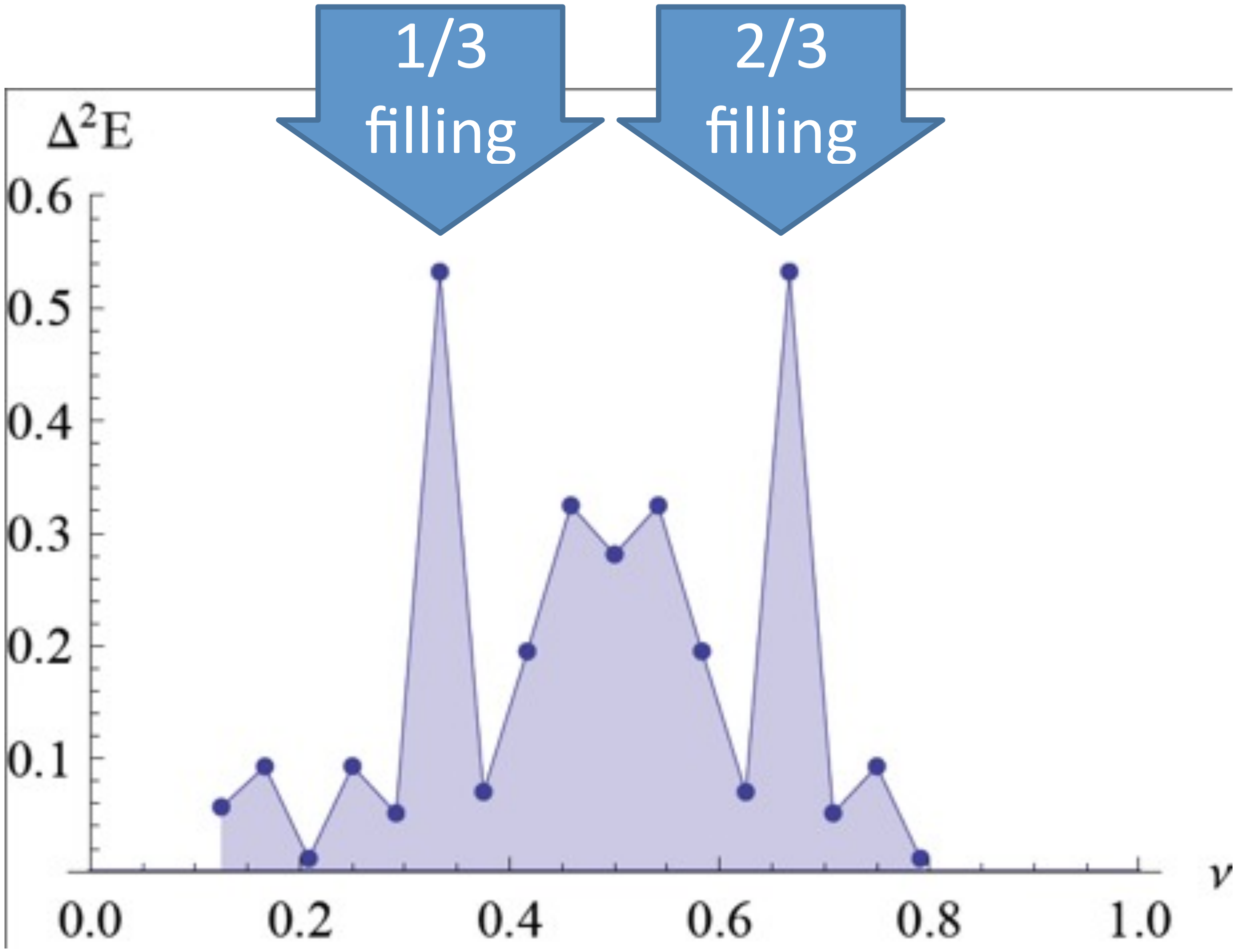


$$H_{\text{int}} := \frac{1}{2}\sum_{i,j} \rho_i V_{i,j} \rho_j \equiv V \sum_{\langle ij \rangle} \rho_i \rho_j , \qquad V > 0$$

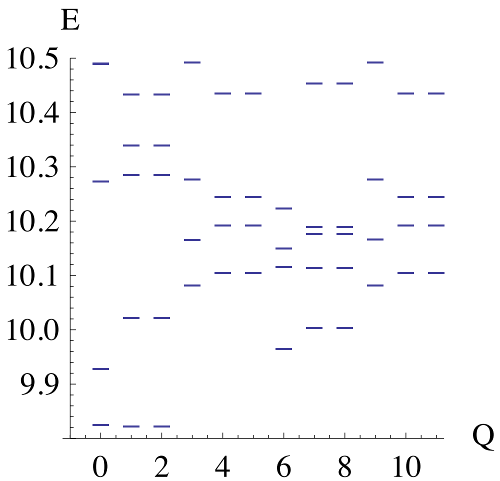
Chern insulator: exact diagonalization

inverse compressibility

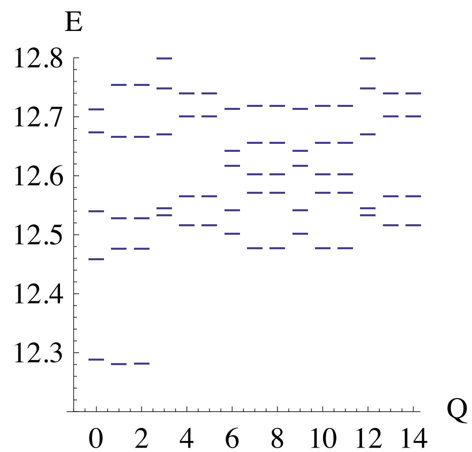
6x4 lattice



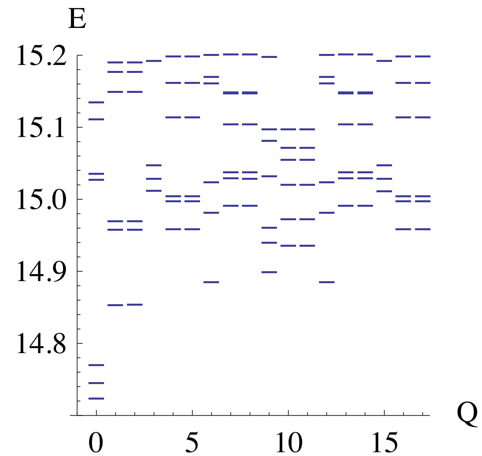
Chern insulator: exact diagonalization



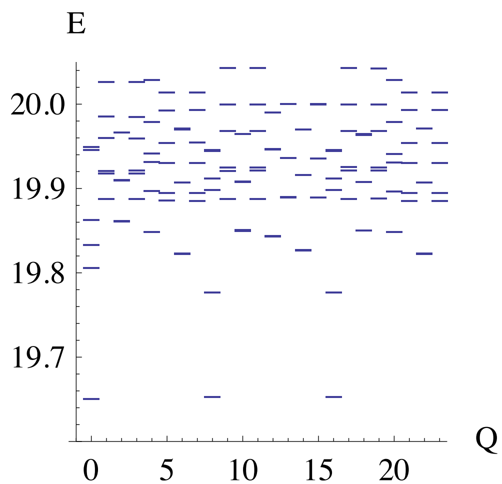
3x4 lattice



3x5 lattice

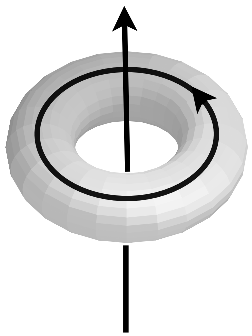


3x6 lattice

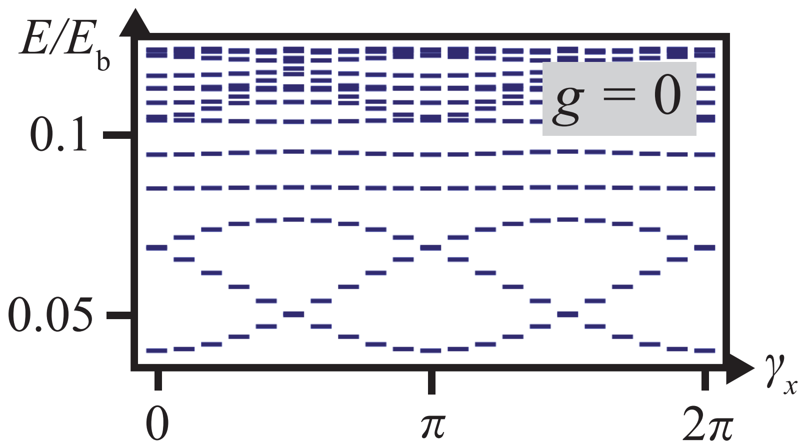


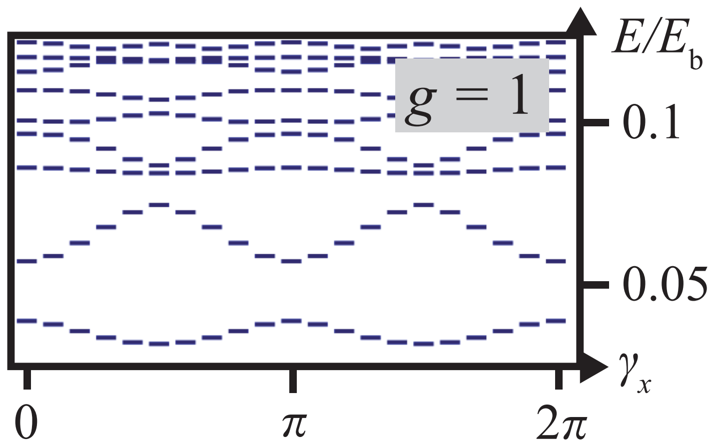
6x4 lattice

Chern insulator: exact diagonalization



3×6 plaquettes (36 sites)





$N_{GS} = 3$

$N_{GS} = 1$

t_2 C = 1

μ C = 0

Is there a fractional Hall effect? Other recent works

- D. N. Sheng, Z. Gu, K. Sun, and L. Sheng, Nature Comm. 2, 389 (2011)
- N. Regnault and B. A. Bernevig, Phys. Rev. X 1, 021014 (2011)
- Yang-Le Wu; B. A. Bernevig, N. Regnault, arXiv:1111.1172
- S. Parameswaran, R. Roy, and S. Sondhi, arXiv:1106.4025
- B. A. Bernevig, N. Regnault arXiv:1110.4488
- M. Goerbig, Eur. Phys. J. B 85(1), 15 (2012)
- R. Shankar and G. Murthy arXiv:1108.5501

Is there a FQHE when flat bands with non-zero Chern number are partially fixed?

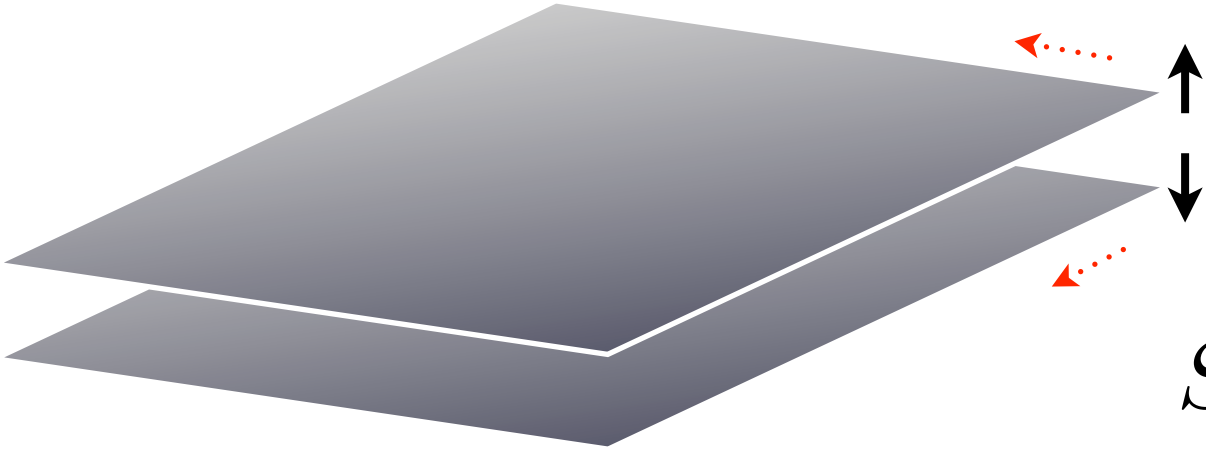
Is there a FQHE when flat bands with non-zero Chern number are partially fixed?

What about time-reversal symmetric systems?

T. Neupert, L. Santos, S. Ryu C. Chamon, and C. Mudry

PRB 84, 165107 (2011)
PRB 84, 165138 (2011)
PRL 108, 046806 (2011)

Spin quantum Hall effect - two FQHE layers



$$S = S_{CS}^{\uparrow} + S_{CS}^{\downarrow}$$

Doubled model with opposite FQHE for each spin species

B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. 96, 106802 (2006).

M. Levin and A. Stern, Phys. Rev. Lett. 103, 196803 (2009).

Time-reversal symmetric Abelian fractional liquids

T. Neupert, L. Santos, S. Ryu C. Chamon, and C. Mudry PRB (2011)

"$\nu = 0$" FQHE

Abelian Chern-Simons action

$$S \equiv \frac{1}{4\pi} \int \mathrm{d}t \, \mathrm{d}^2\mathbf{x} \;\; \epsilon_{\mu\nu\rho} \, K_{ij} \, a^i_\mu \, \partial_\nu \, a^j_\rho$$

constraints from TRS

$$K = \begin{pmatrix} \kappa & \Delta \\ \Delta^{\mathsf{T}} & -\kappa \end{pmatrix}$$

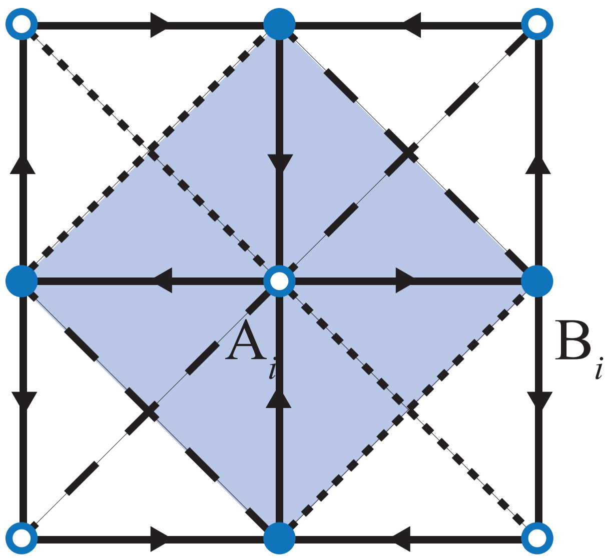
$$\kappa^{\mathsf{T}} = \kappa, \quad \Delta^{\mathsf{T}} = -\Delta$$

degeneracy on torus
for this subclass

$$\begin{aligned} \#_{\mathrm{GS}} &= \left| \det \begin{pmatrix} \kappa & \Delta \\ \Delta^{\mathsf{T}} & -\kappa \end{pmatrix} \right| = \left| \det \begin{pmatrix} \Delta^{\mathsf{T}} & -\kappa \\ \kappa & \Delta \end{pmatrix} \right| \\ &= \left[\mathrm{Pf} \begin{pmatrix} \Delta^{\mathsf{T}} & -\kappa \\ \kappa & \Delta \end{pmatrix} \right]^2 = [\mathrm{integer}]^2 \end{aligned}$$

Lattice realization of time-reversal symmetric fractional topological liquids

T. Neupert, L. Santos, S. Ryu C. Chamon, and C. Mudry PRB (2011); PRL (2011).



2 copies of flatband models,
with opposite chirality for ↑↓ spins

<u>flattened Kane+Mele model</u>

Add interactions

$$H_{\text{int}} := U \sum_{i\in\Lambda} \rho_{i,\uparrow}\rho_{i,\downarrow} + V \sum_{\langle ij\rangle\in\Lambda} \left(\rho_{i,\uparrow}\rho_{j,\uparrow} + \rho_{i,\downarrow}\rho_{j,\downarrow} + 2\lambda\rho_{i,\uparrow}\rho_{j,\downarrow}\right)$$

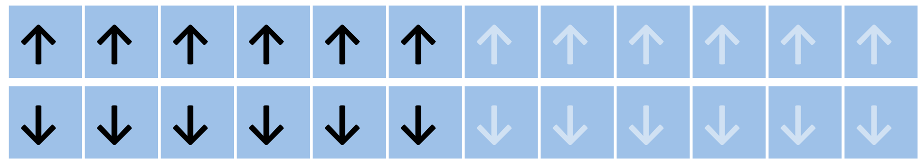
Topological insulator: 1/2 filling

$$H := \sum_{\boldsymbol{k}\in \mathrm{BZ}} c^{\dagger}_{\boldsymbol{k}}\, \mathcal{H}_{\boldsymbol{k}}\, c_{\boldsymbol{k}} + U \sum_{\boldsymbol{r}} \sum_{\alpha=A,B} n_{\boldsymbol{r},\uparrow,\alpha} n_{\boldsymbol{r},\downarrow,\alpha}$$

1/2 filling of lower bands,

4x3 lattice,

12 particles



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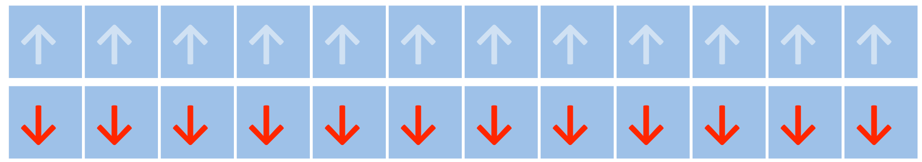

1/2 filling of lower bands,
4x3 lattice,
12 particles

flat band
ferromagnetism

Topological insulator: 1/2 filling

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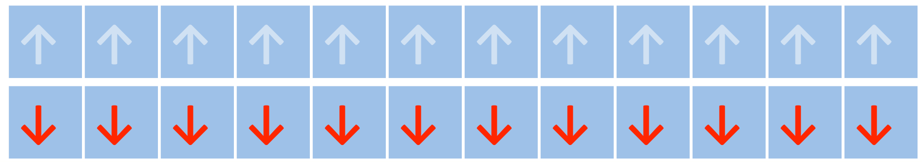
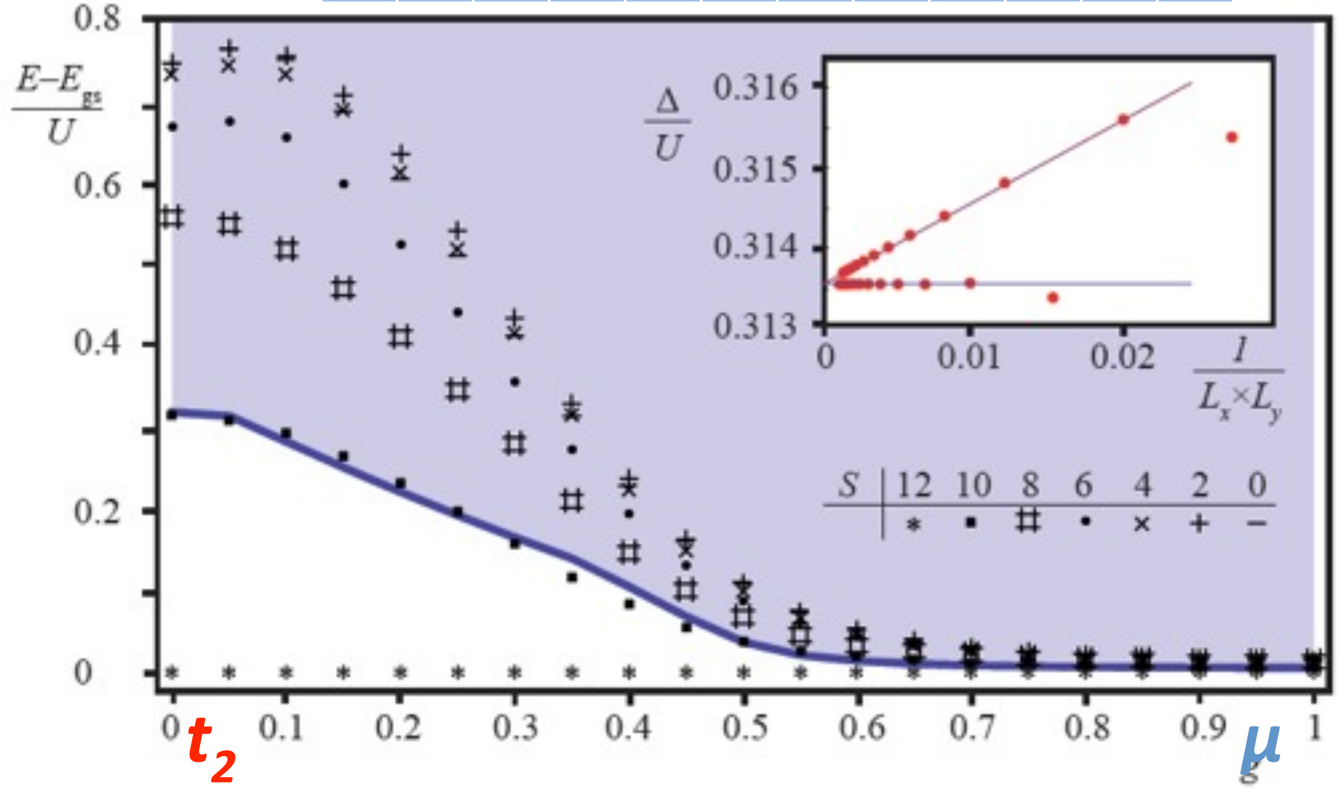
1/2 filling of lower bands,
4x3 lattice,
12 particles



flat band
ferromagnetism



Topological insulator: 1/2 filling

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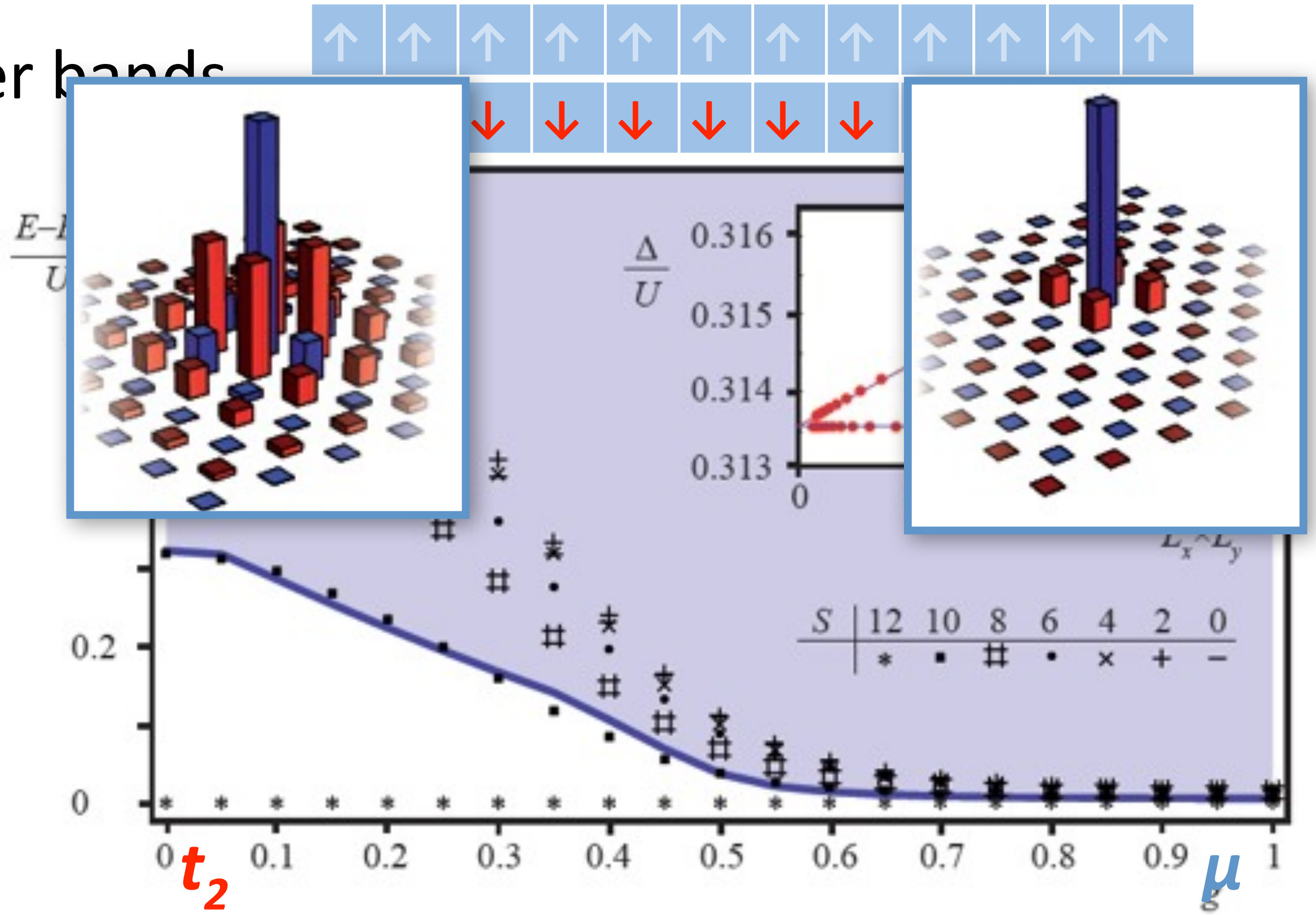
1/2 filling of lower bands

4x3 lattice,

12 particles

flat band

ferromagnetism



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1/2 filling of lower bands

4x3 lattice,

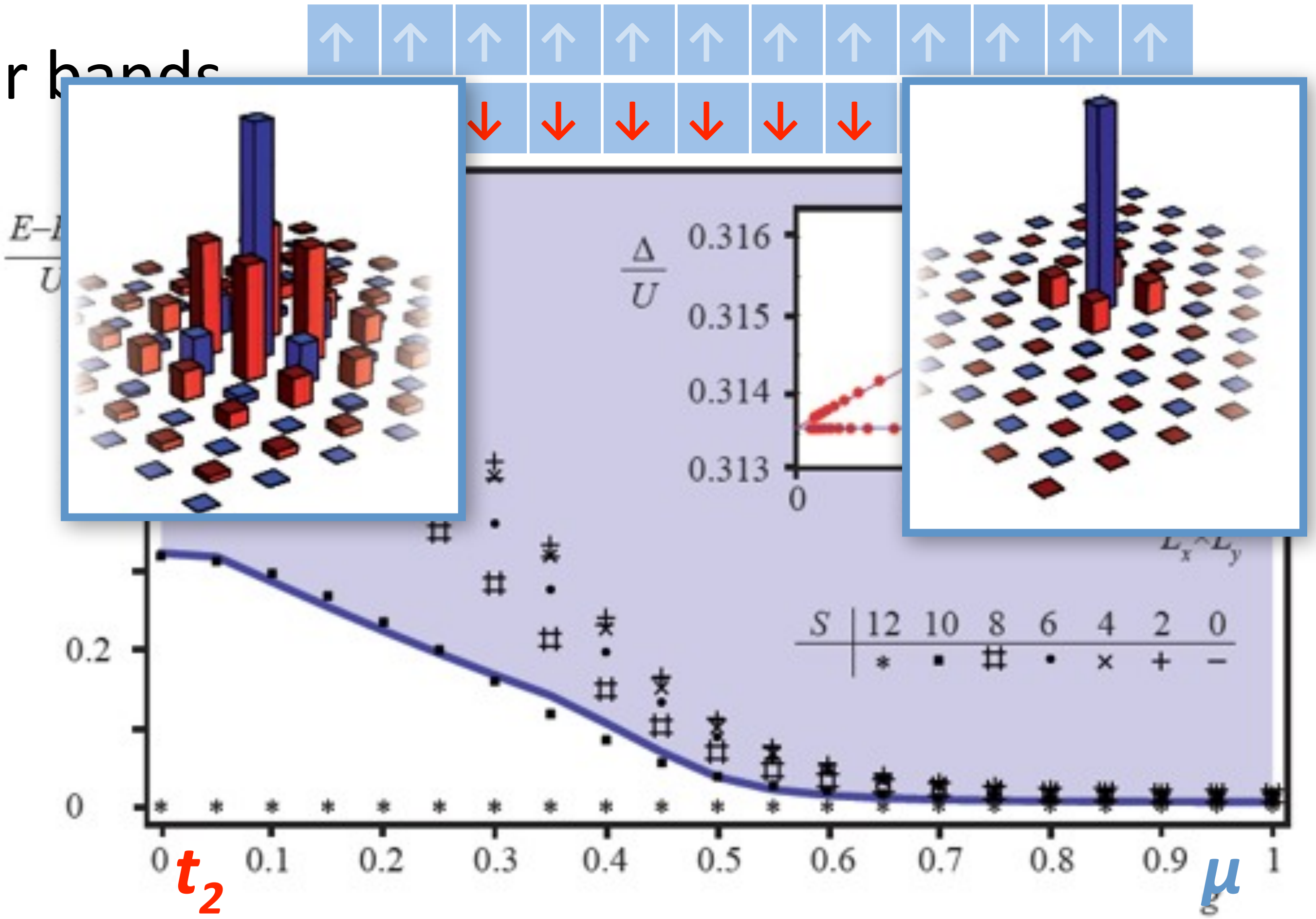
12 particles

flat band

ferromagnetism

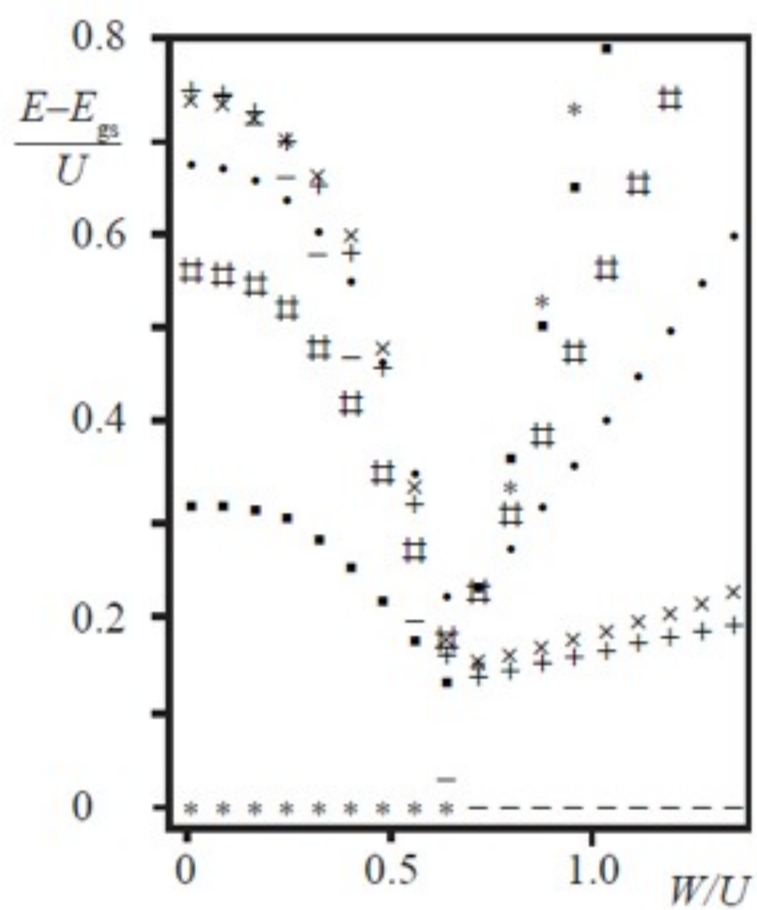
Spontaneous

QHE $\sigma_{xy} = \frac{e^2}{h}$

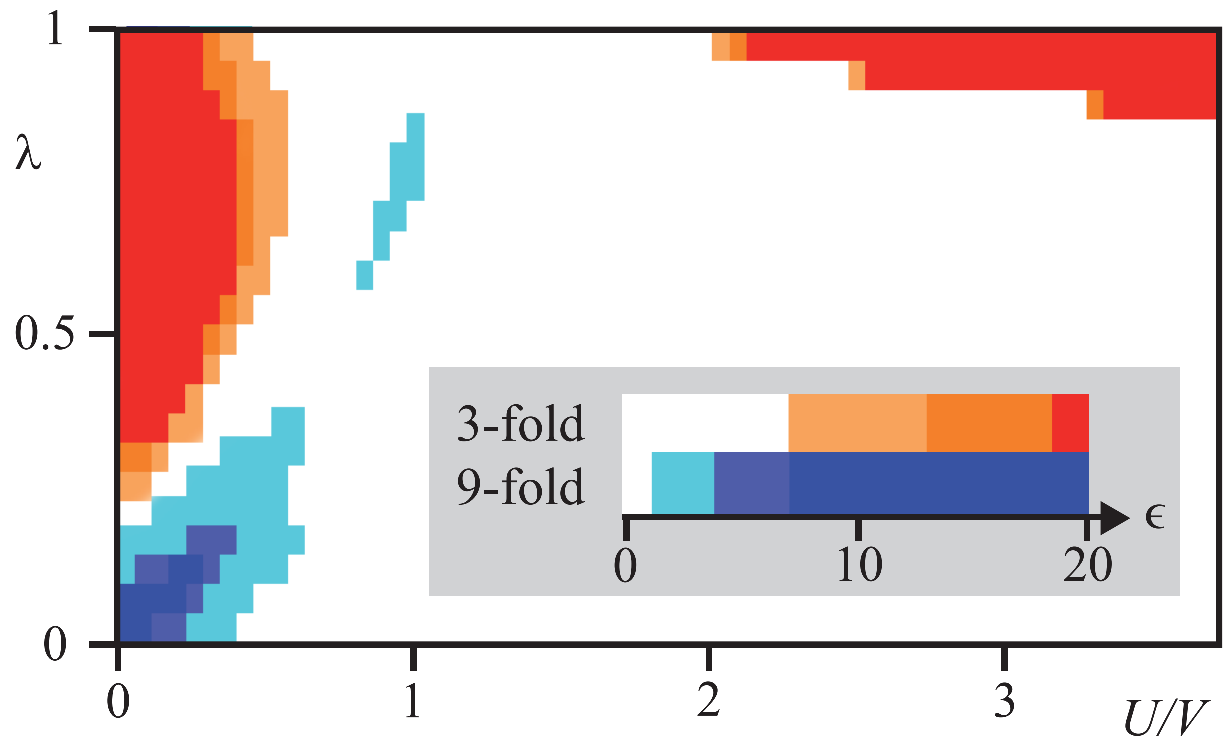


Topological insulator: 1/2 filling

Restoring bandwidth:
full polarization stable up to W ≈ 0.7 U

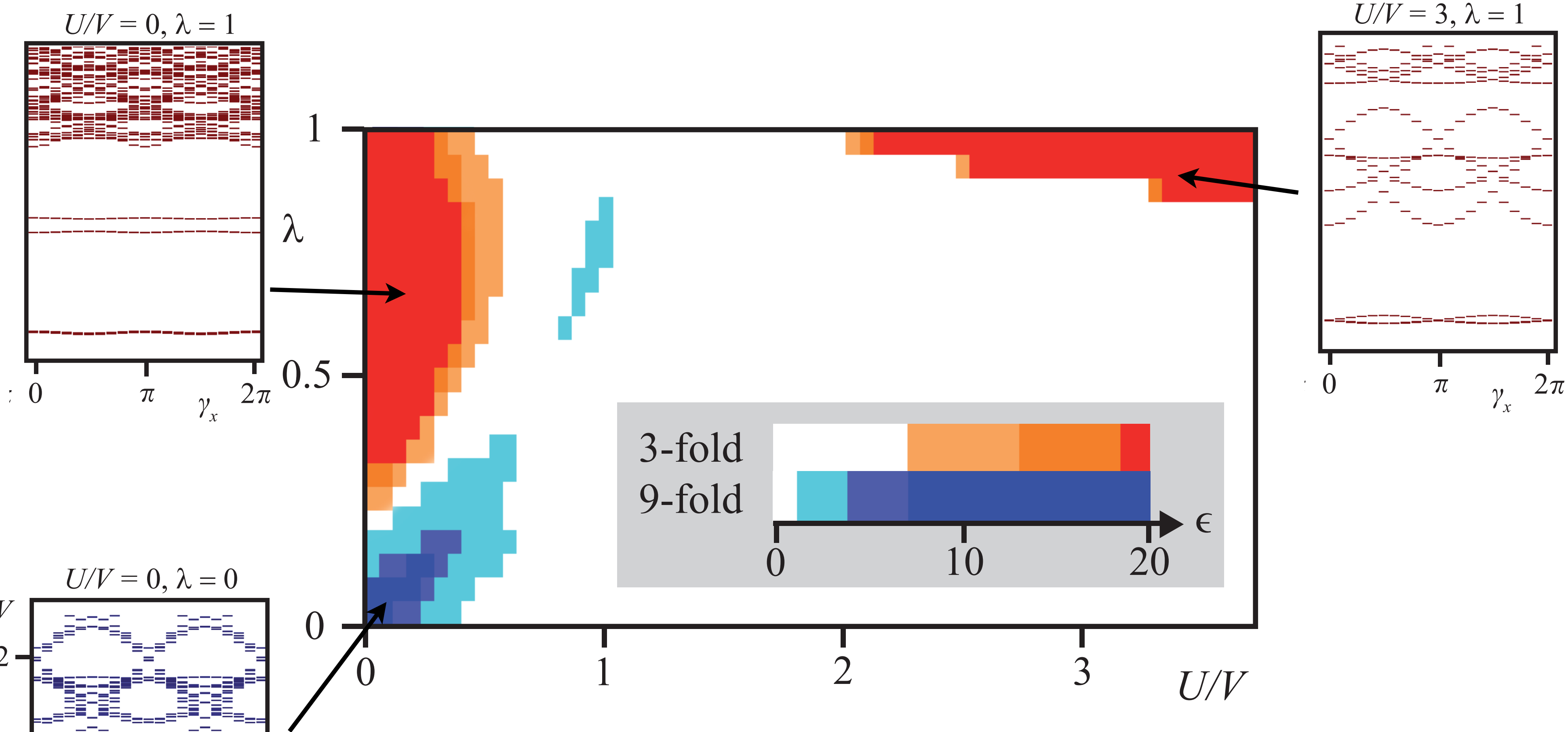


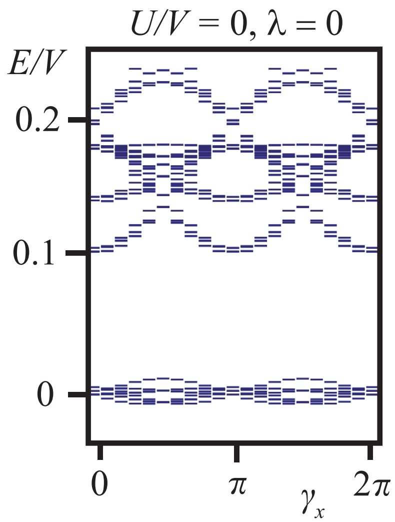
Topological insulator: 2/3 filling



$$H_{\text{int}} := U \sum_{i\in\Lambda} \rho_{i,\uparrow}\rho_{i,\downarrow} + V \sum_{\langle ij\rangle\in\Lambda} \left(\rho_{i,\uparrow}\rho_{j,\uparrow} + \rho_{i,\downarrow}\rho_{j,\downarrow} + 2\lambda\rho_{i,\uparrow}\rho_{j,\downarrow}\right)$$

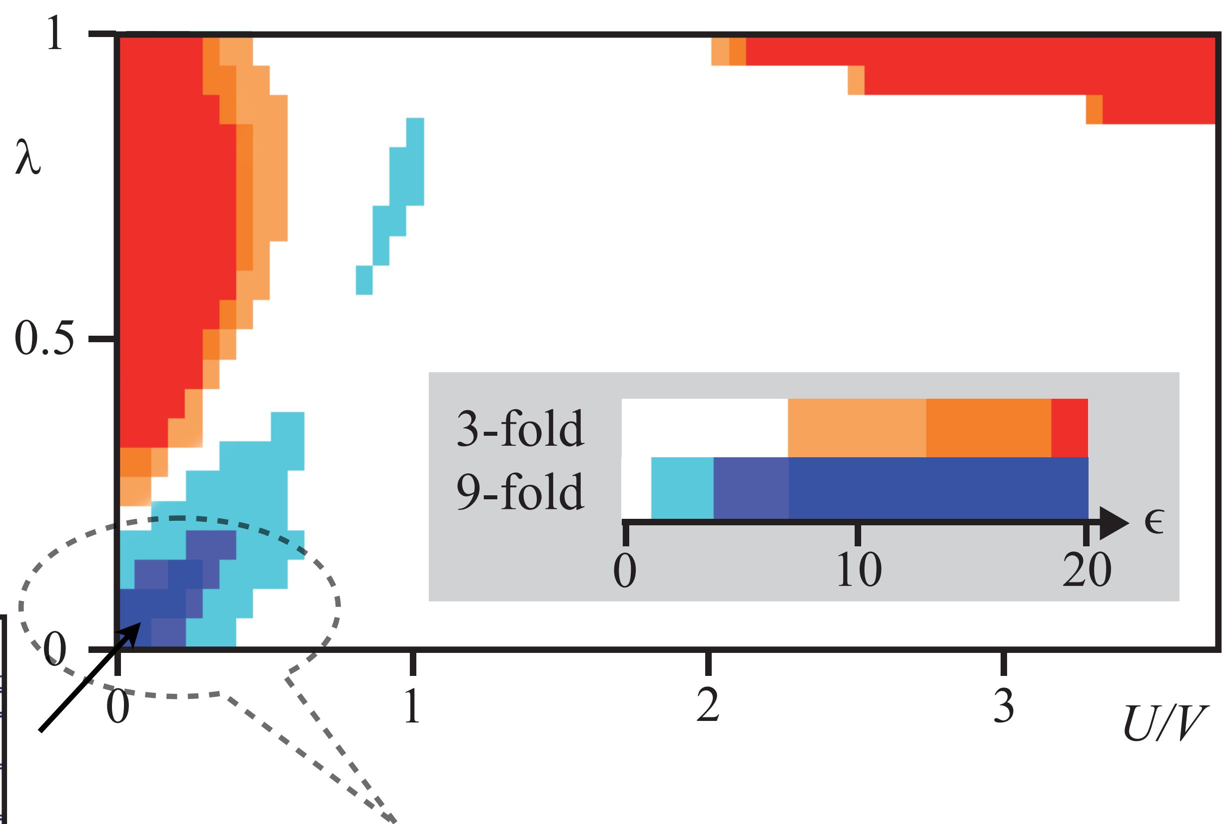
Topological insulator: 2/3 filling

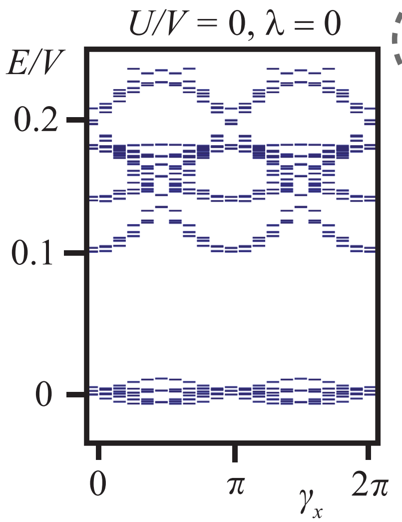




$$H_{\mathrm{int}} := U \sum_{i \in \Lambda} \rho_{i,\uparrow}\rho_{i,\downarrow} + V \sum_{\langle ij \rangle \in \Lambda} \left(\rho_{i,\uparrow}\rho_{j,\uparrow} + \rho_{i,\downarrow}\rho_{j,\downarrow} + 2\lambda \rho_{i,\uparrow}\rho_{j,\downarrow} \right)$$

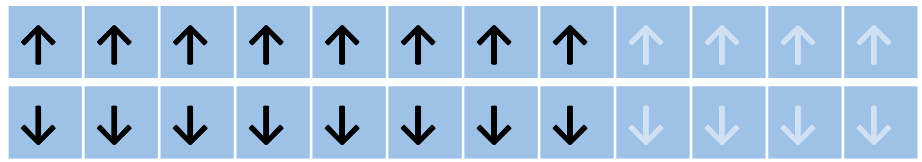
Topological insulator: 2/3 filling

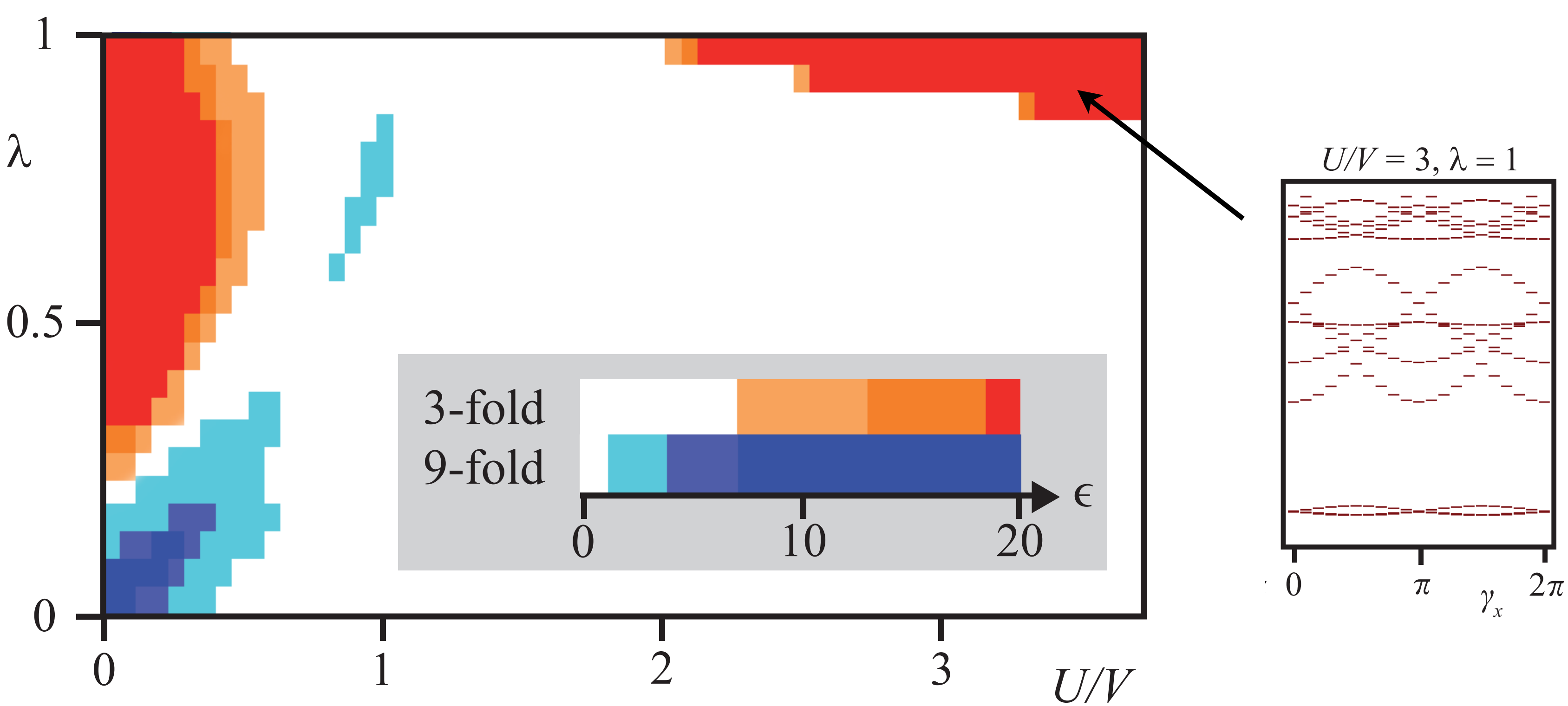




lattice realization of a $\Delta = 0$ (Bernevig+Zhang+Levin+Stern)
TRS fractional topological insulator

Topological insulator: 2/3 filling

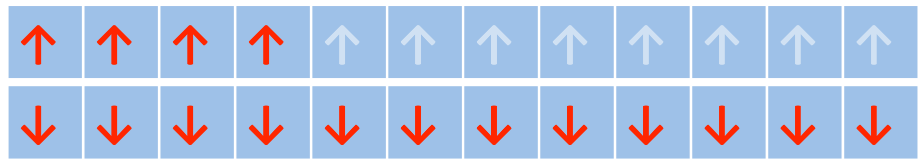




Topological insulator: 2/3 filling

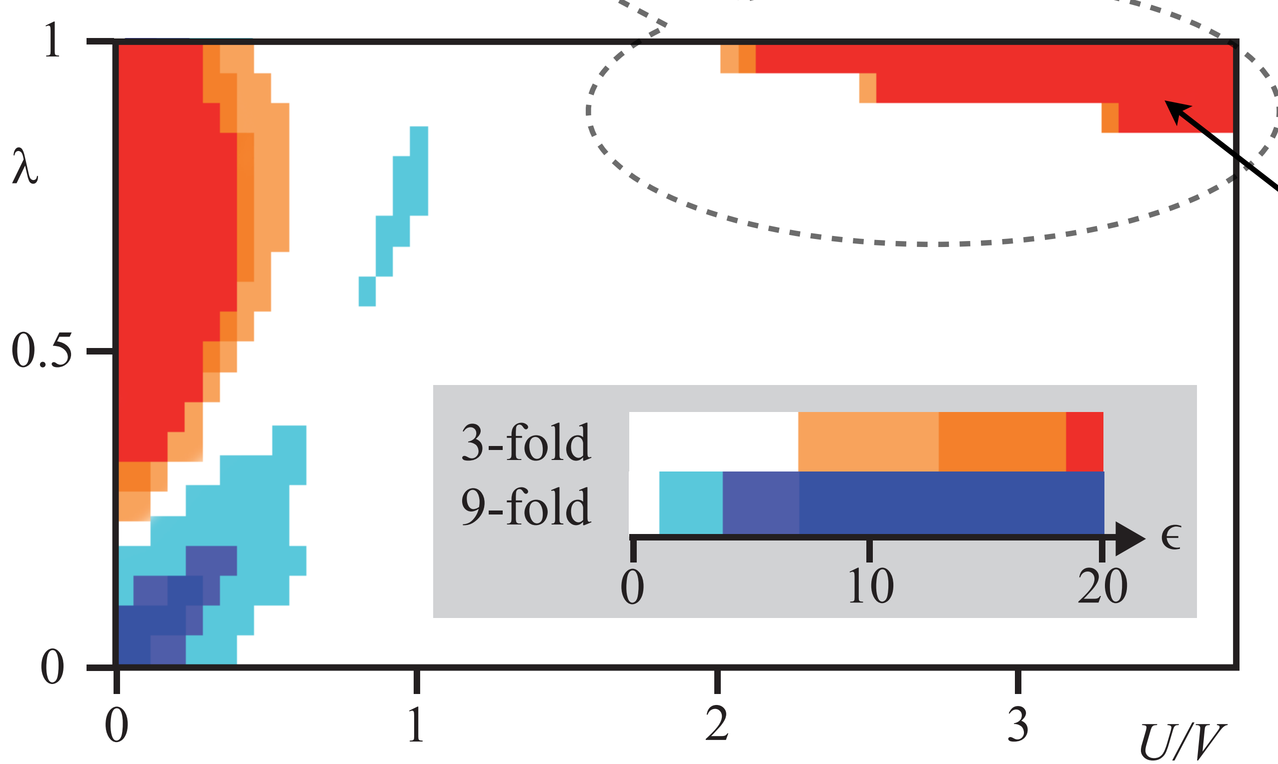
spontaneous TRS breaking

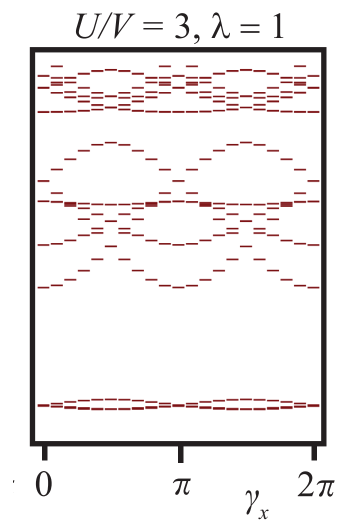
spontaneous FQHE



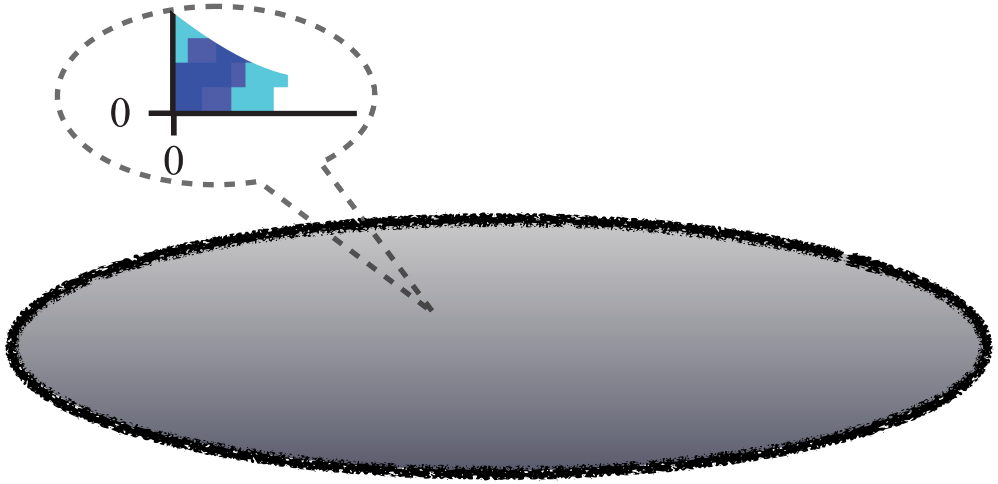
$\nu_\uparrow = 1/3$

$\nu_\downarrow = 1$





Perspective - bulk vs. edge



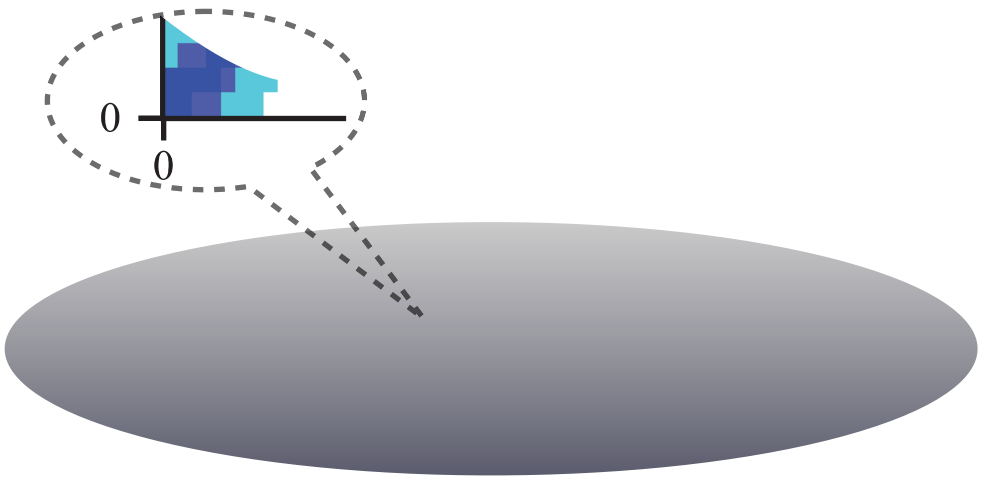
Edge modes?

M. Levin and A. Stern, PRL (2009)

T. Neupert et al. PRB (2011)

$$K = \begin{pmatrix} +1 & +2 & 0 & 0 \\ +2 & +1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

Perspective - bulk vs. edge



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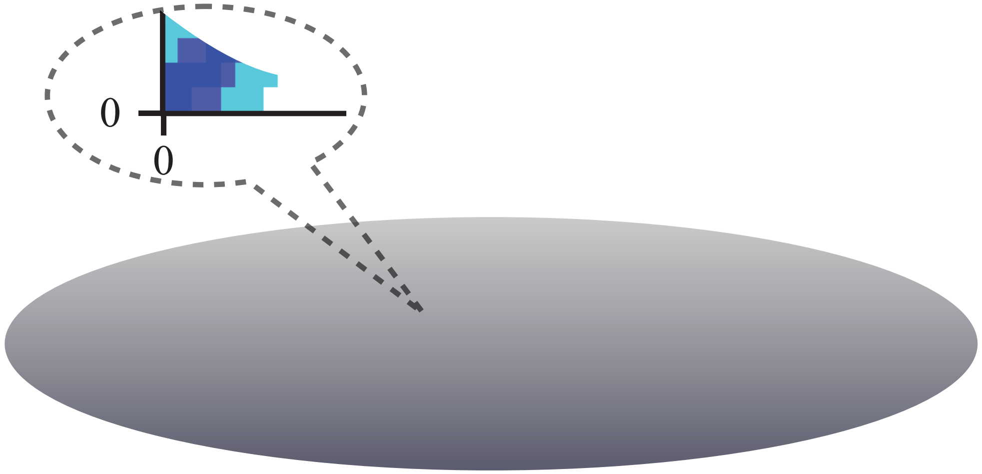

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No propagating edge modes, but...
fractionalized excitations in the bulk!
Bulk rich ... edge poor

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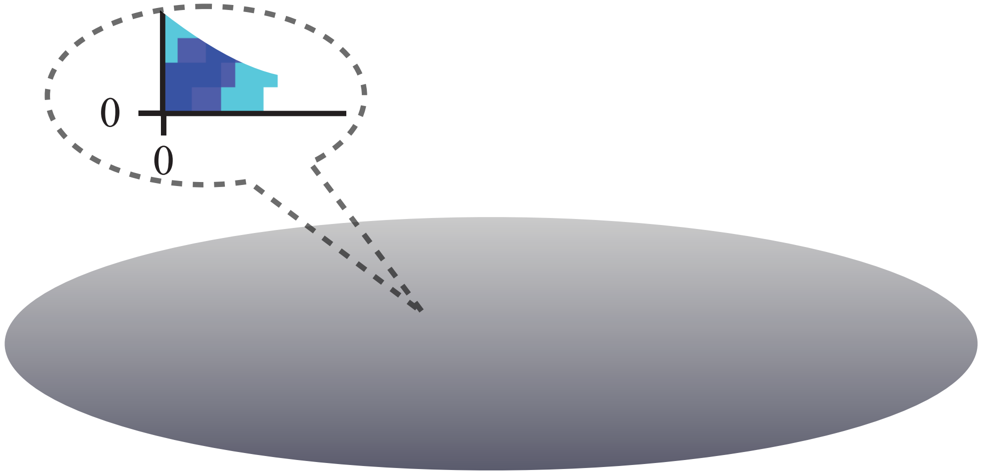

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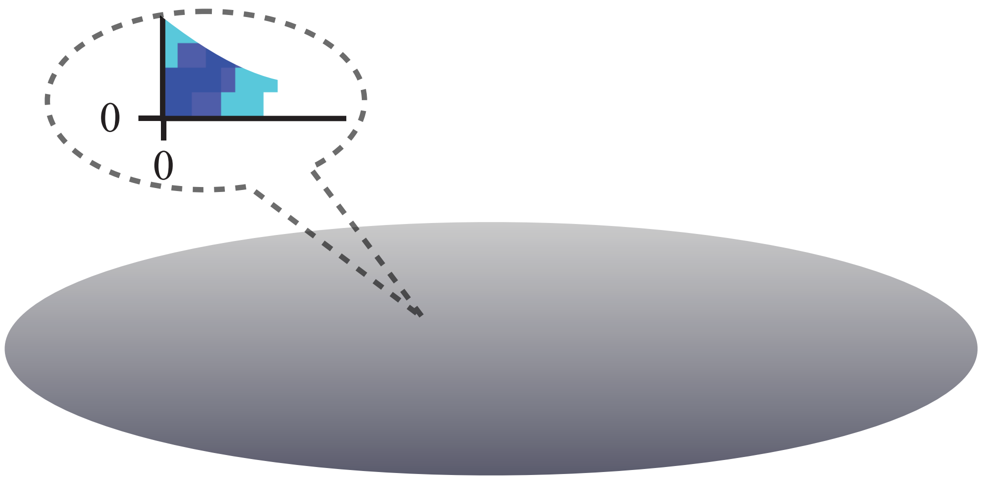

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3D

T. Neupert, L. Santos, S. Ryu C. Chamon, and C. Mudry arXiv:1202.5188

Classification of non-interacting states of matter
“Periodic Table of Topological Insulators”

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997) (AZ) and are specified by presence or absence of $\mathcal{T}$ symmetry Θ, particle-hole symmetry Ξ and chiral symmetry $\Pi = \Xi\Theta$. ± 1 and 0 denotes the presence and absence of symmetry, with ± 1 specifying the value of Θ^2 and Ξ^2. As a function of symmetry and space dimensionality, d, the topological classifications ($\mathbb{Z}$, $\mathbb{Z}_2$ and 0) show a regular pattern that repeats when $d \to d + 8$.

M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010)

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TI

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chiral

TI

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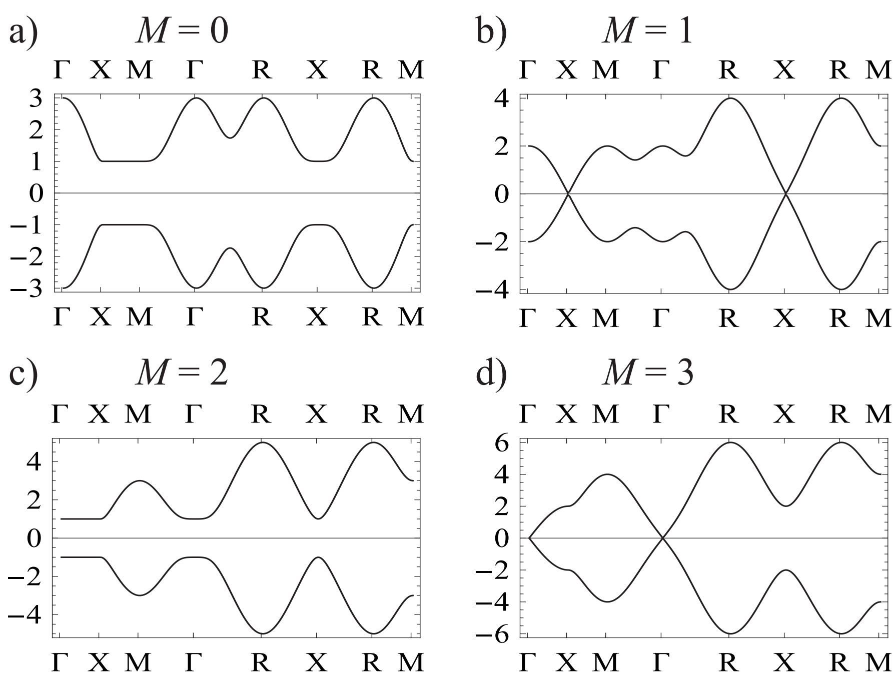
3D Lattice model

3 orbital model on the cubic lattice

$$H_{\mathbf{k}} = \sum_{i=1}^{3} \lambda_{3+i} \sin k_i + \lambda_7 \left(M - \sum_{i=1}^{3} \cos k_i \right)$$

Gell-Mann matrices

flat band



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Gell-Mann matrices

flat band

$$H_{\mathbf{k}} = \begin{pmatrix} 0_{2\times 2} & q_{\mathbf{k}} \\ q_{\mathbf{k}}^{\dagger} & 0 \end{pmatrix}$$

chiral class (AIII)

3D Lattice model

3 orbital model on the cubic lattice

$$H_{\mathbf{k}} = \sum_{i=1}^{3} \lambda_{3+i} \sin k_i + \lambda_7 \left(M - \sum_{i=1}^{3} \cos k_i \right)$$

$$\theta(M) = \begin{cases} +2\pi, & |M| < 1, \\ -\pi, & 1 < |M| < 3, \\ 0, & 3 < |M|. \end{cases}$$

Non-commuting projected position operators

2D: Projection onto Landau level

$$\widehat{\mathbf{X}} = \widehat{\mathcal{P}}_n \, \widehat{\mathbf{R}} \, \widehat{\mathcal{P}}_n$$

$$= \widehat{\mathbf{R}} - \frac{\ell_B^2}{\hbar} \mathbf{e_3} \times \widehat{\mathbf{\Pi}}$$

$$\left[\widehat{X}_1 , \widehat{X}_2 \right] = +\mathrm{i} \, \ell_B^2$$

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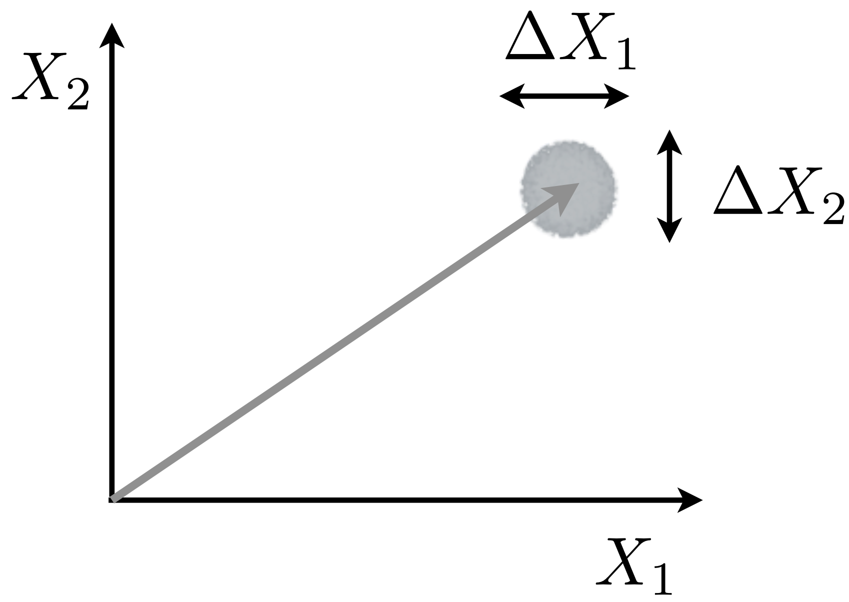

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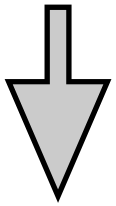
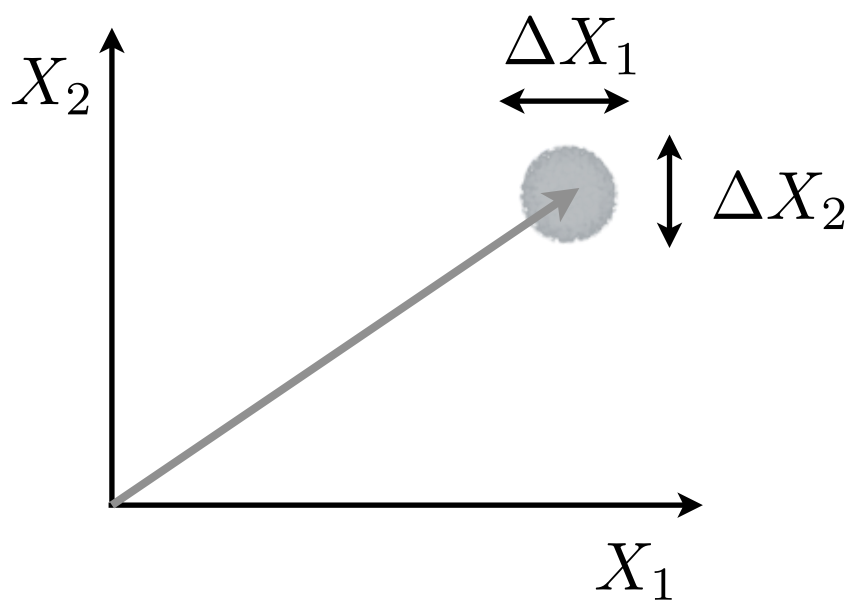
non-commuting coordinates



"fuzziness"



Non-commuting projected position operators

2D: Coordinate transformation

$$r_\mu \to f_\mu(\mathbf{r})$$

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Poisson bracket $$\{f_1, f_2\}_{\mathrm{P}}\,(\mathbf{r}) = \epsilon^{\mu\nu}\left(\frac{\partial \mathbf{f_1}}{\partial \mathbf{r}_\mu}\frac{\partial \mathbf{f_2}}{\partial \mathbf{r}_\nu}\right)$$

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transformation preserves area $\Rightarrow$ commutators are unchanged

Non-commuting projected position operators

3D: what would be a “pristine” extension?

$$\left[\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\right] = +\mathrm{i}\,\ell^3$$

Nambu (quantum) 3-bracket

$$[\widehat{A}_1, \widehat{A}_2, \widehat{A}_3] = \epsilon_{ijk}\, \widehat{A}_i \widehat{A}_j \widehat{A}_k$$

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$$\{f_1, f_2, f_3\}_{\mathrm{N}}\,(\mathbf{r}) = \epsilon^{\mu\nu\lambda}\left(\frac{\partial \mathbf{f_1}}{\partial \mathbf{r}_\mu}\frac{\partial \mathbf{f_2}}{\partial \mathbf{r}_\nu}\frac{\partial \mathbf{f_3}}{\partial \mathbf{r}_\lambda}\right)(\mathbf{r})$$

Nambu (classical) 3-bracket

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$$\{f_1, f_2, f_3\}_{\mathrm{N}}\,(\mathbf{r}) = \epsilon^{\mu\nu\lambda} \left(\frac{\partial \mathbf{f_1}}{\partial \mathbf{r}_\mu} \frac{\partial \mathbf{f_2}}{\partial \mathbf{r}_\nu} \frac{\partial \mathbf{f_3}}{\partial \mathbf{r}_\lambda}\right)(\mathbf{r})$$

Nambu (classical) 3-bracket

transformation preserves volume $\Longrightarrow$ 3-brackets are unchanged

Non-commuting projected position operators

3D: non-pristine

$$\left[\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\right] = +\mathrm{i}\,\ell^3 + \ldots$$

$$\widehat{X}_\mu = \int \mathrm{d}^d\mathbf{k}\ \widehat{\chi}^\dagger_{\tilde{\mathbf{a}}}(\mathbf{k})\ \mathrm{i}\,\mathbf{D}^{\tilde{\mathbf{a}}\tilde{\mathbf{b}}}_\mu(\mathbf{k})\ \widehat{\chi}_{\tilde{\mathbf{b}}}(\mathbf{k})$$

$$D_\mu(\mathbf{k}) = \partial_\mu + \mathbf{A}_\mu(\mathbf{k})$$

when $\quad \mathrm{Tr}\left[D_\mu, D_\nu\right] = V\int \mathrm{d}^d\mathbf{k}\ \mathrm{tr}\,\mathbf{F}_{\mu\nu}(\mathbf{k}) = \mathbf{0}$

$$\left\langle : \left[\widehat{X}_\mu, \widehat{X}_\nu, \widehat{X}_\rho\right] : \right\rangle = -\,\mathrm{i}\,\mathrm{Tr}\left[D_\mu, D_\nu, D_\rho\right]$$

Non-commuting projected position operators

3D: non-pristine

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A. P. Polychronakos (2007)

Non-commuting projected position operators

3D: non-pristine

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A. P. Polychronakos (2007)

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$$\bar{\rho} = \frac{N_\mathrm{p}}{V}$$

$$\mathrm{CS}^{(3)} = \frac{\pi}{2} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ \epsilon^{ijk}\ \mathrm{tr}\left(A_i F_{jk} - \frac{2}{3} A_i A_j A_k \right)$$

Non-commuting projected position operators

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$$\Big\langle : \Big[\widehat{X}_\mu, \widehat{X}_\nu, \widehat{X}_\rho\Big] : \Big\rangle = -\,\mathrm{i}\,\mathrm{Tr}\,\Big[D_\mu, D_\nu, D_\rho\Big]$$

A. P. Polychronakos (2007)

$$\frac{1}{N_{\mathrm{p}}}\Big\langle : \Big[\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\Big] : \Big\rangle = \frac{12\pi^2\,\mathrm{i}}{\bar{\rho}}\,\mathrm{CS}^{(3)}$$

$$\frac{1}{N_{\mathrm{p}}}\Big\langle : \Big[\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\Big] : \Big\rangle = +\mathrm{i}\ell^3$$

Non-commuting projected position operators

3D: fuzzy coordinates

$$\frac{1}{N_{\mathrm{p}}}\left\langle : \left[\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\right] : \right\rangle = +\mathrm{i}\ell^3$$

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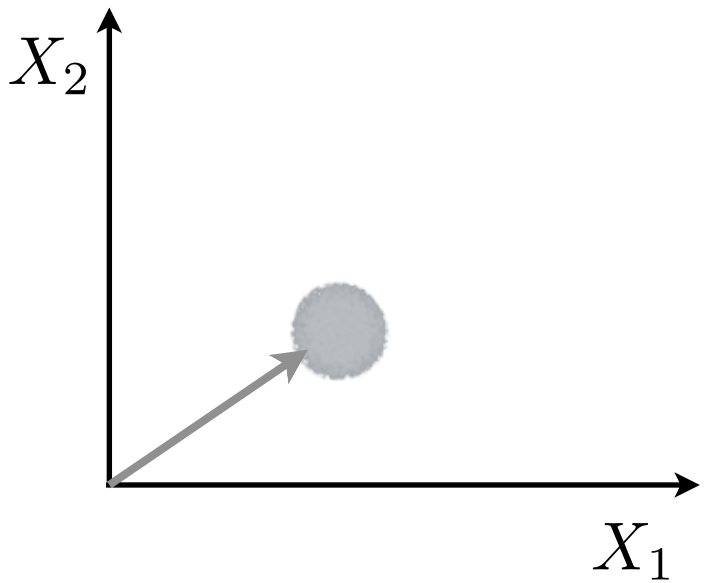
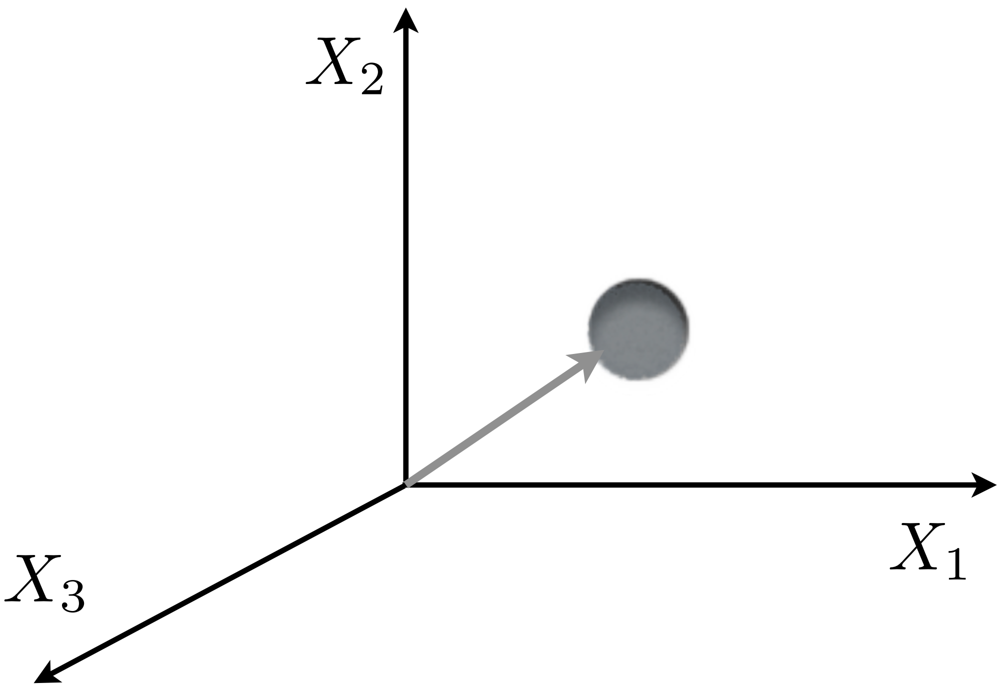


Non-commuting projected position operators

3D: fuzzy coordinates

$$\frac{1}{N_{\mathrm{p}}}\left\langle :\left[\widehat{X}_1,\widehat{X}_2,\widehat{X}_3\right]:\right\rangle = +\mathrm{i}\ell^3$$



Summary

It is possible to have dispersionless bands (flat bands) that are isolated from all others by an energy gap.

The isolated flat bands can be non-trivial in the sense of having a non-zero Chern number, and thus sustain an IQHE when fully occupied.

When partially filled, electron-electron interactions can lead to the FQHE and other topological phases in time-reversal invariant systems.

Topological Hubbard models have to ferromagnetic ground states: symmetry breaking simultaneously with IQHE or FQHE.

Fractional TI in 3D: non-commuting coordinates