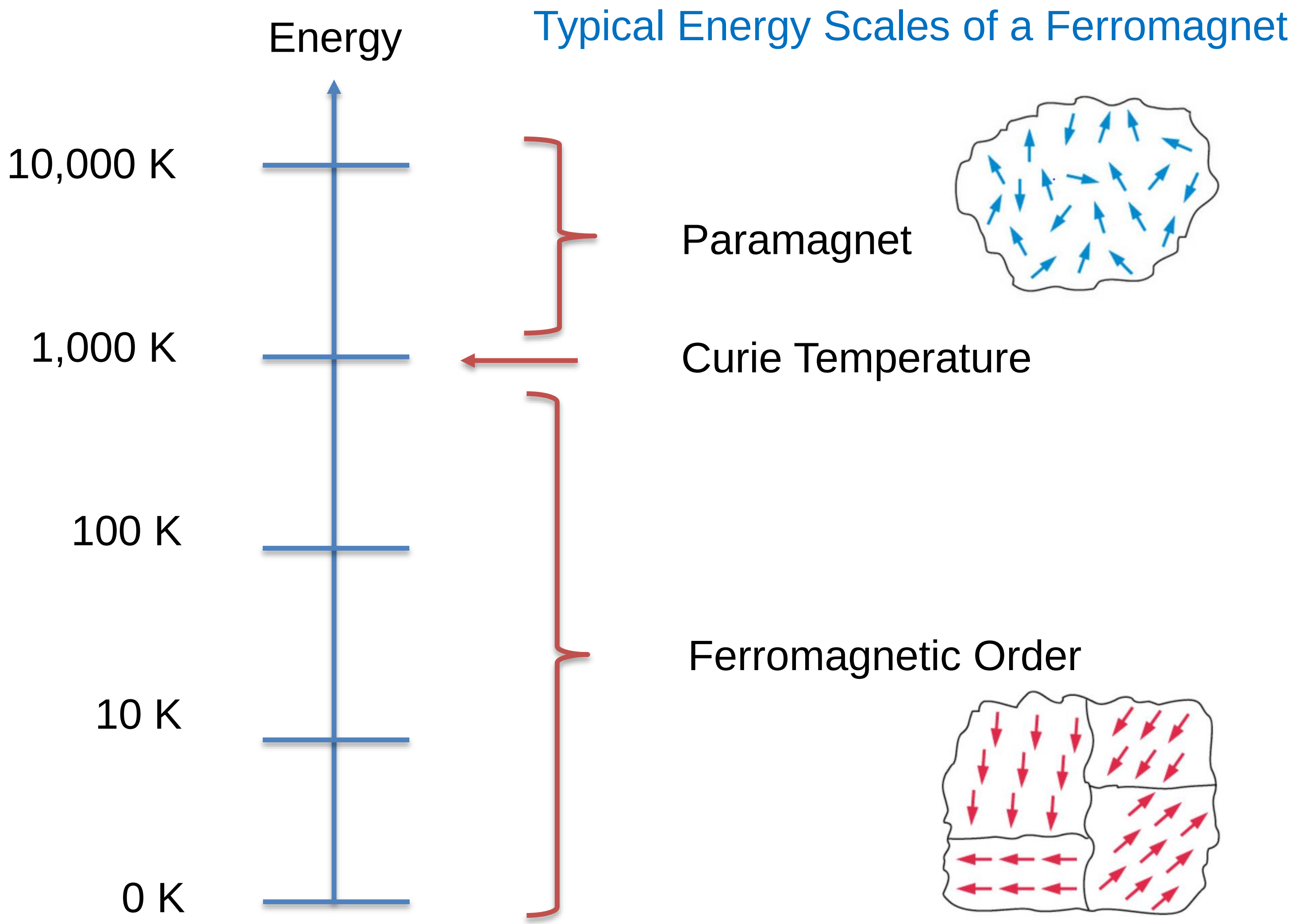
The Bulk-Edge Correspondence for Abelian Quantum Hall States

Michael Mulligan
Station Q/UCSB

In collaboration with Jennifer Cano, Meng Cheng, Michael Freedman, Chetan Nayak, Eugeniu Plamadeala, and Jon Yard

(i) Phys. Rev. B 88, 045131 (2013)
(ii) ArXiv: 1310.5708
(iii) Work in Progress

Physics is Organized by Scale

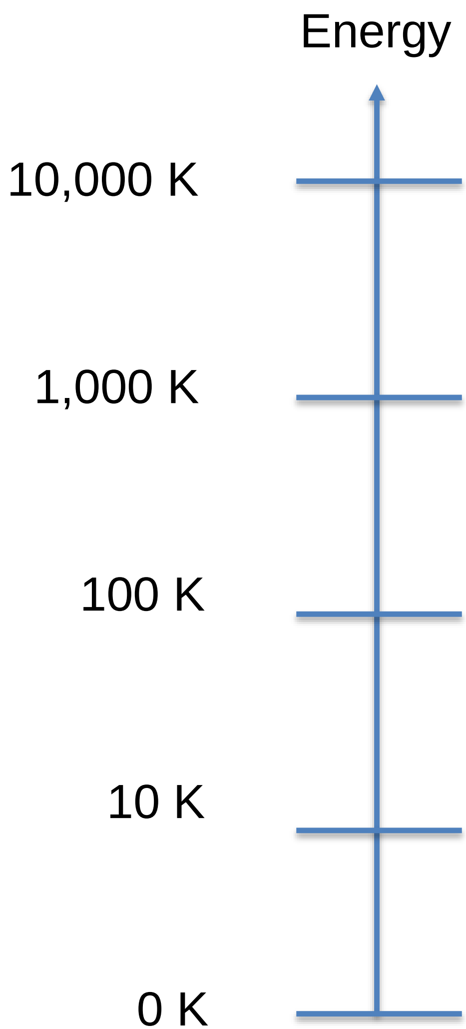


The Mother of All Effective Hamiltonians

$$H_{UV} = \sum_i^{\mathrm{N_{electrons}}} \frac{\hat{P}_i^2}{2m_{\mathrm{electron}}} + \sum_j^{\mathrm{N_{protons}}} \frac{\hat{P}_j^2}{2m_{\mathrm{proton}}} + V_{\mathrm{electron-electron}} + V_{\mathrm{electron-proton}} + \vec{E}^2 + \vec{B}^2$$

$$N_{\mathrm{electrons}} \sim N_{\mathrm{protons}} \sim 10^{23}$$

Effective Hamiltonians



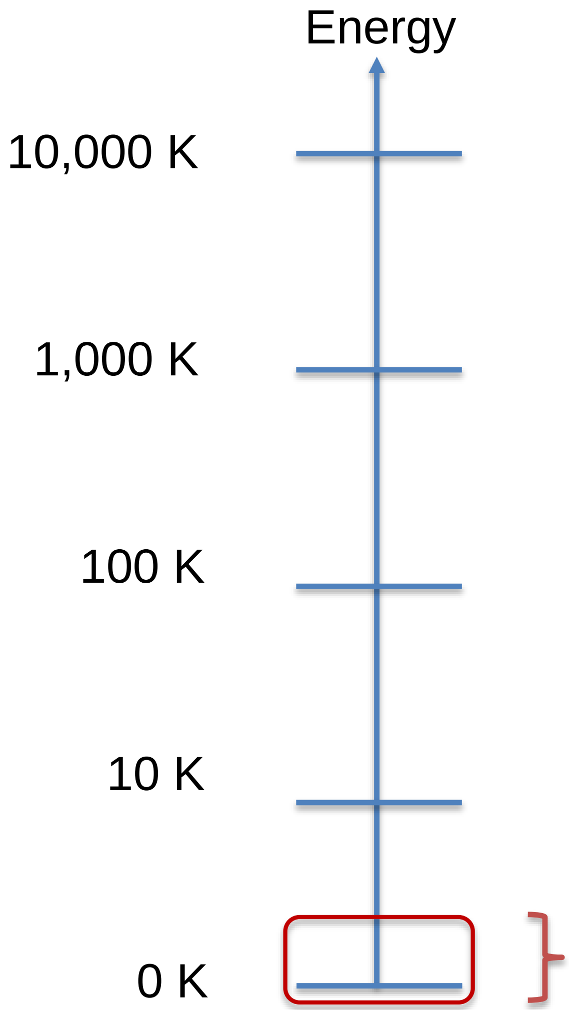
Ferromagnets

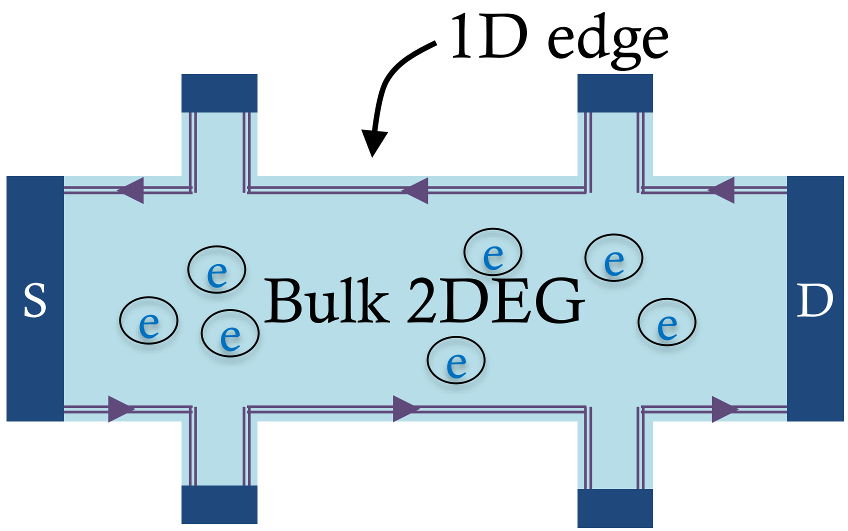
$$H_{UV}$$

$$H_{IR} = -J \sum_{<i,j>} S_i \cdot S_j$$

Semiconductor Heterostructures can be Surprising:

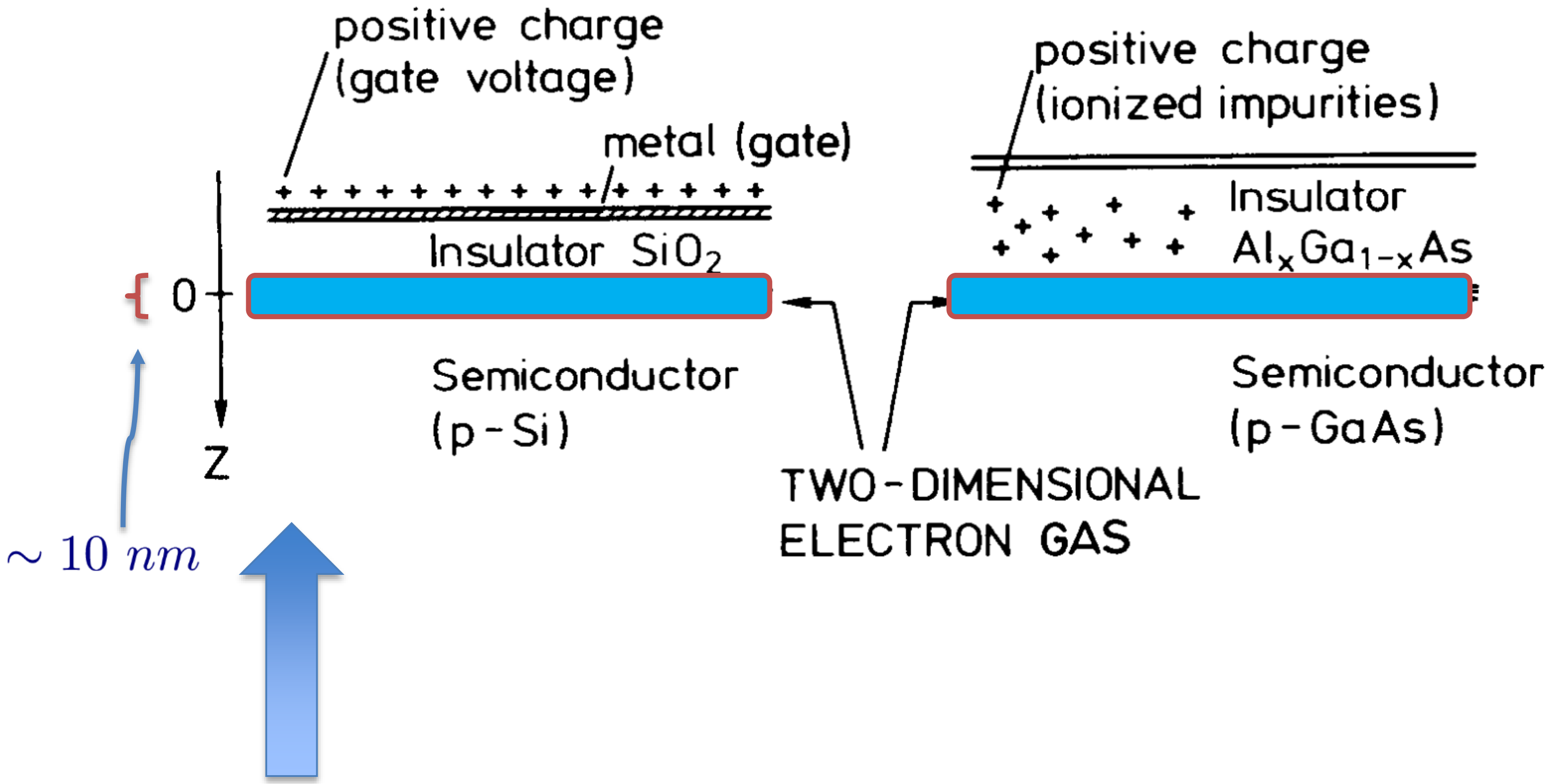
Quantum Hall Effect





- low temperatures < 1 K
- high magnetic fields > 1-20 T
- density ~ $10^{11} - 10^{12}\ cm^{-2}$

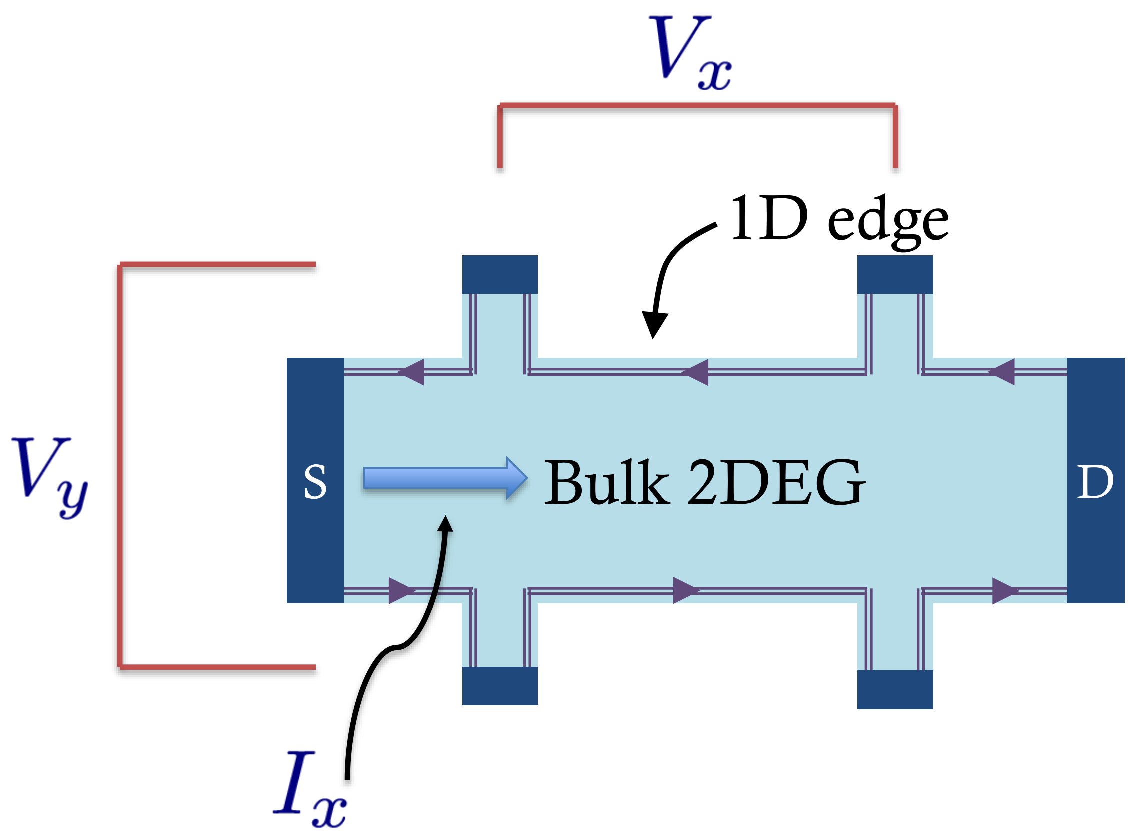
2D Electron Gas in a B Field



Magnetic Field

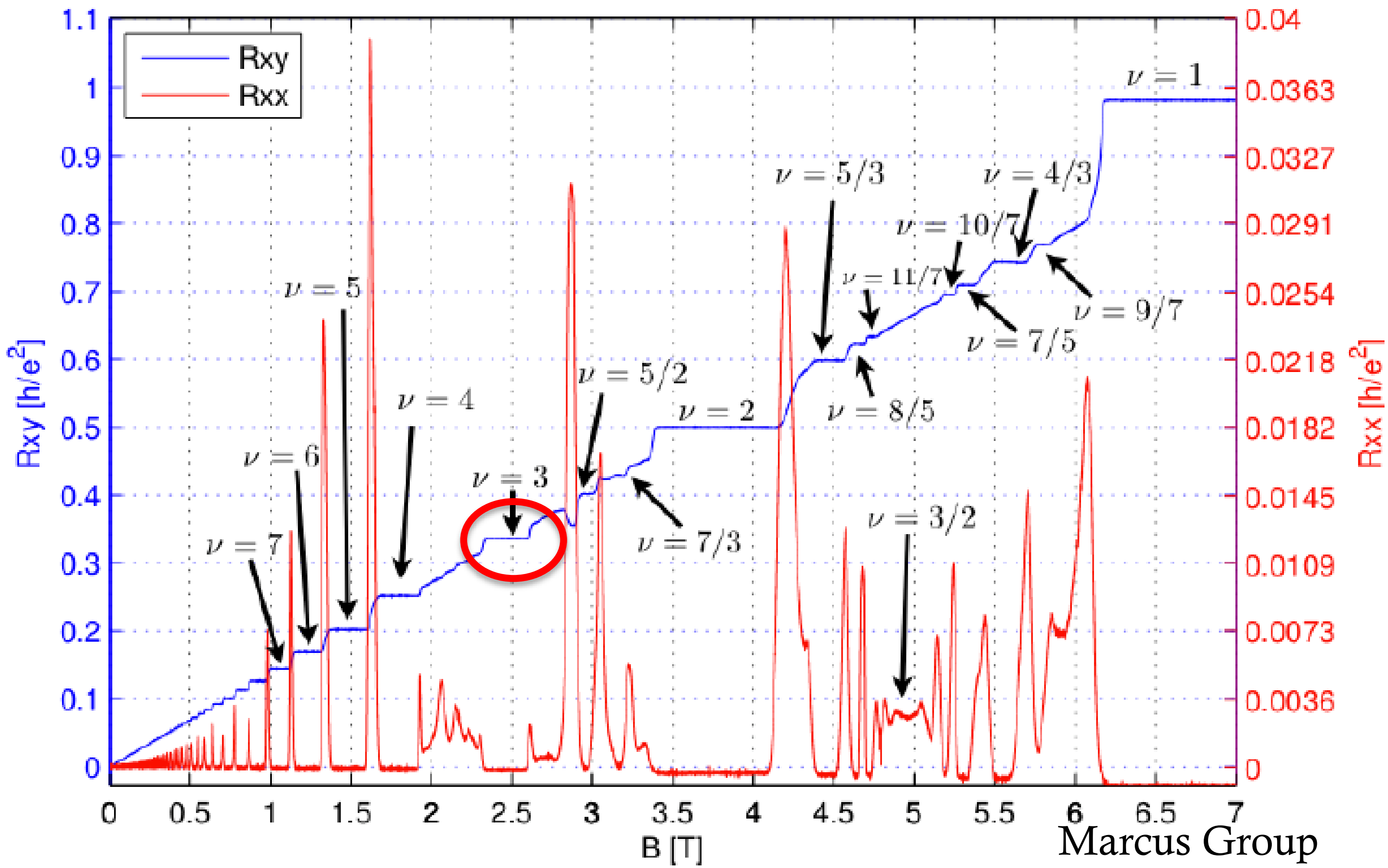
Von Klitzing, Dorda, & Pepper

Hall and Longitudinal Resistances



$$R_{xy} = \frac{V_y}{I_x} \qquad R_{xx} = \frac{V_x}{I_x}$$

Quantum Hall Effect

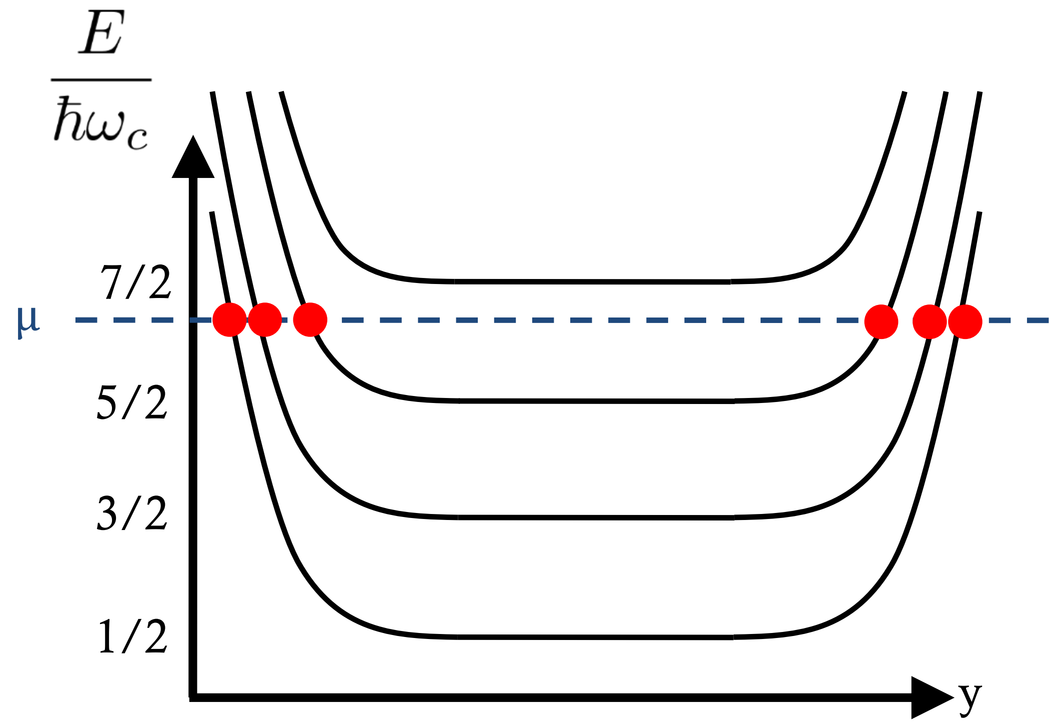


Marcus Group

$$R_{xy} = \nu^{-1}\frac{h}{e^2} \qquad R_{xx} = R_{yy} = 0$$

Integer Quantum Hall Edge Modes

Example: $\upsilon = 3$



<u>Gapless excitations</u>

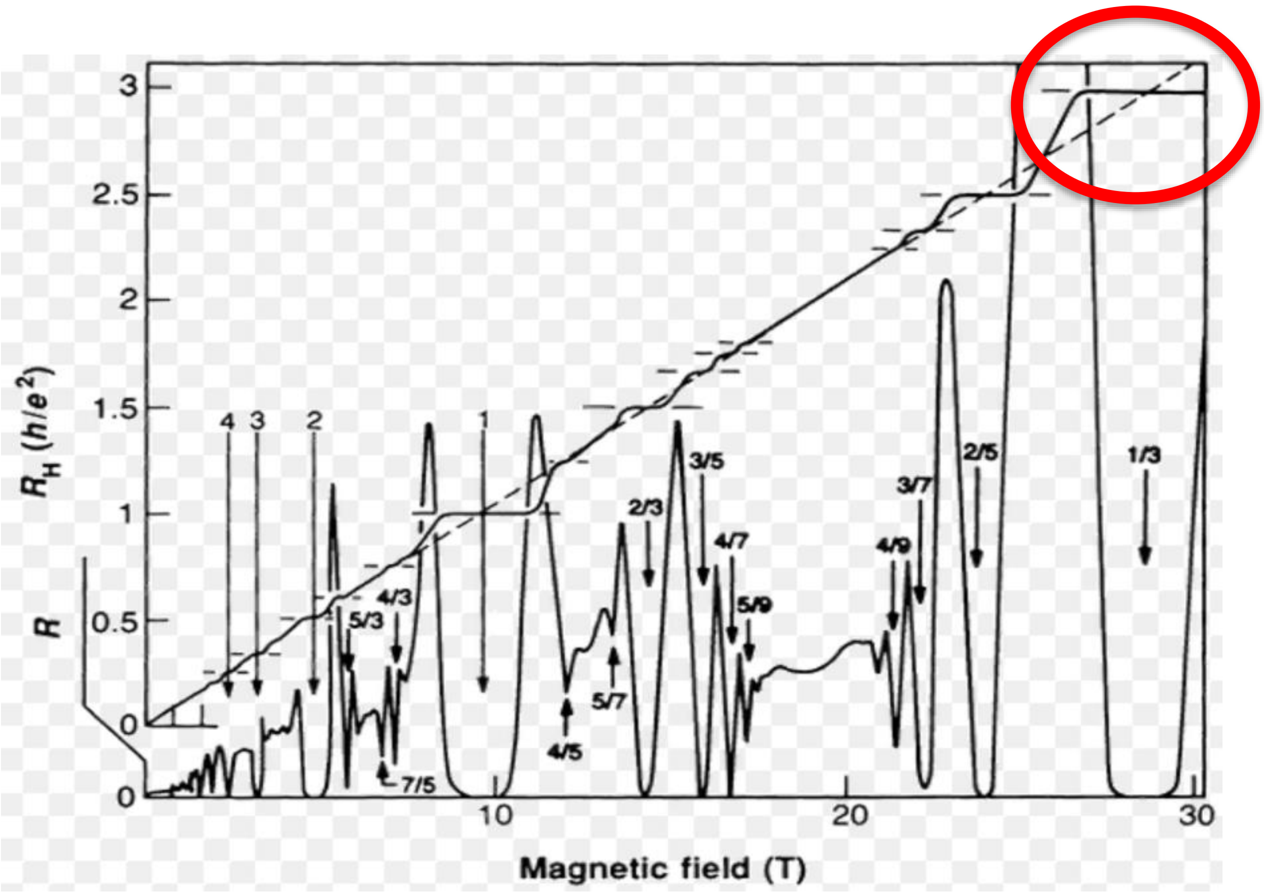
1. Fermionic
2. Near the edge
3. One per filled Landau level
4. Linear dispersion
5. Chiral $y \propto k_x l_B^2$

Landau levels bend up near the edge of a sample and intersect the chemical potential

Halperin 1982

Fractional Quantum Hall Effect

Example: $\upsilon = 1/3$

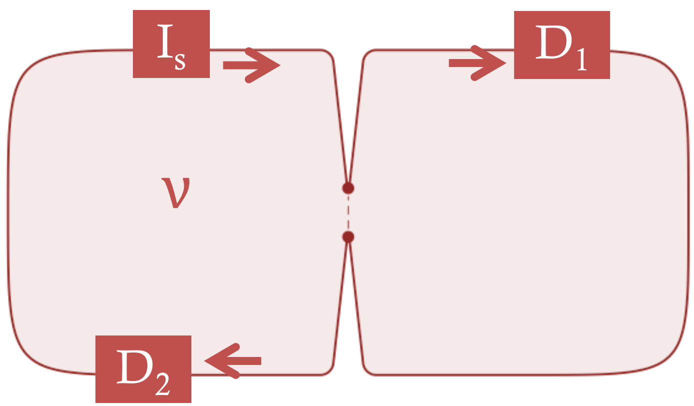


Stormer and Tsui

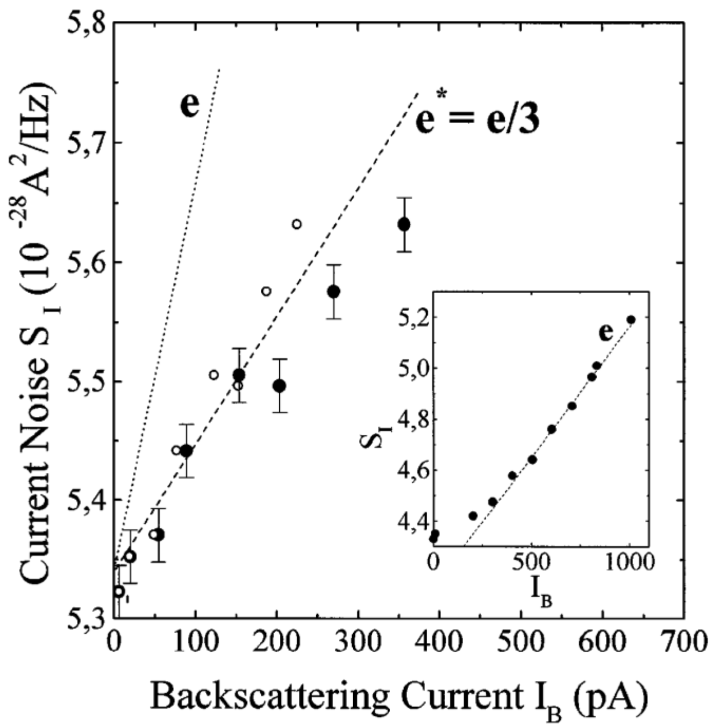
Topological Order and Experimental Signatures

1. Fractionalization of Charge: e/3 Quasiparticles

Observed in Shot Noise or current fluctuation Measurements



$$\text{Shot Noise} = 2(\frac{e}{3})I_B$$



Saminadayar, Glattli, Jin and Etienne

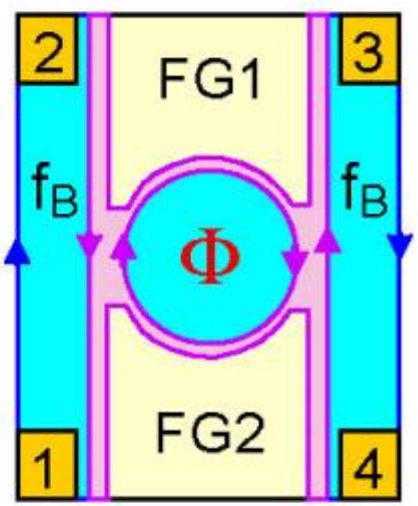
Theory: Kane-Fisher; Fendley-Ludwig-Saleur

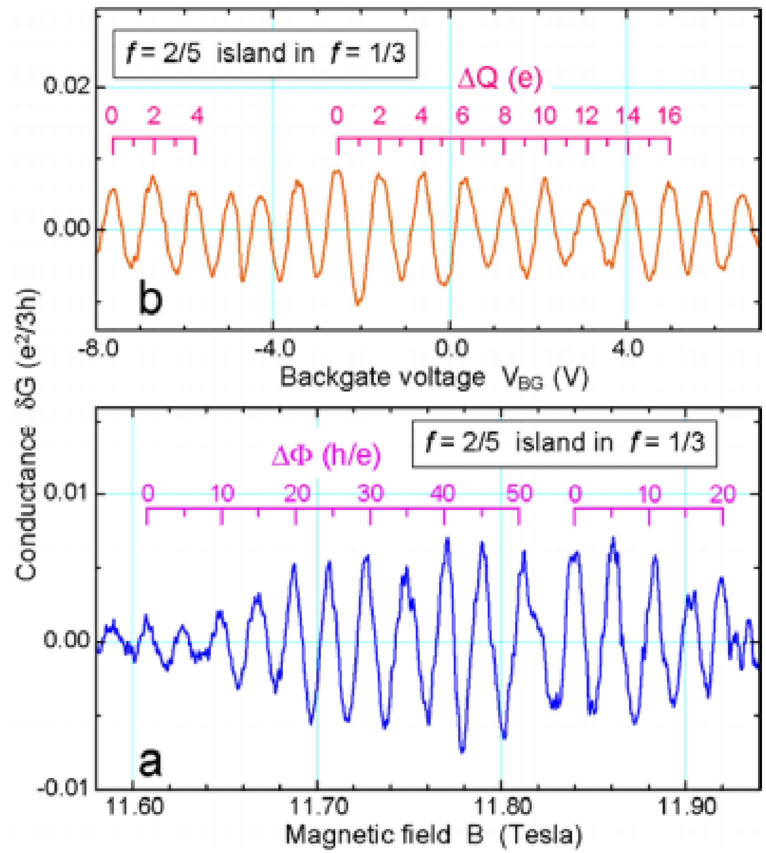
Experimental Signatures

1. Anyonic Statistics – generalization of Bose-Fermi statistics

$$\psi(x_1, x_2, \ldots) \rightarrow e^{\frac{\pi i}{3}} \psi(x_2, x_1, \ldots)$$

'Indications'/'encouragement' of its observance in interferometry





Camino, Zhou, and Goldman

Numerical Signatures

Topology-dependent ground state degeneracy

Not observable in actual experiments as you need to fabricate a torus in the lab

Useful in numerics

($\mathbf{Z}_4$ top order)

Wen

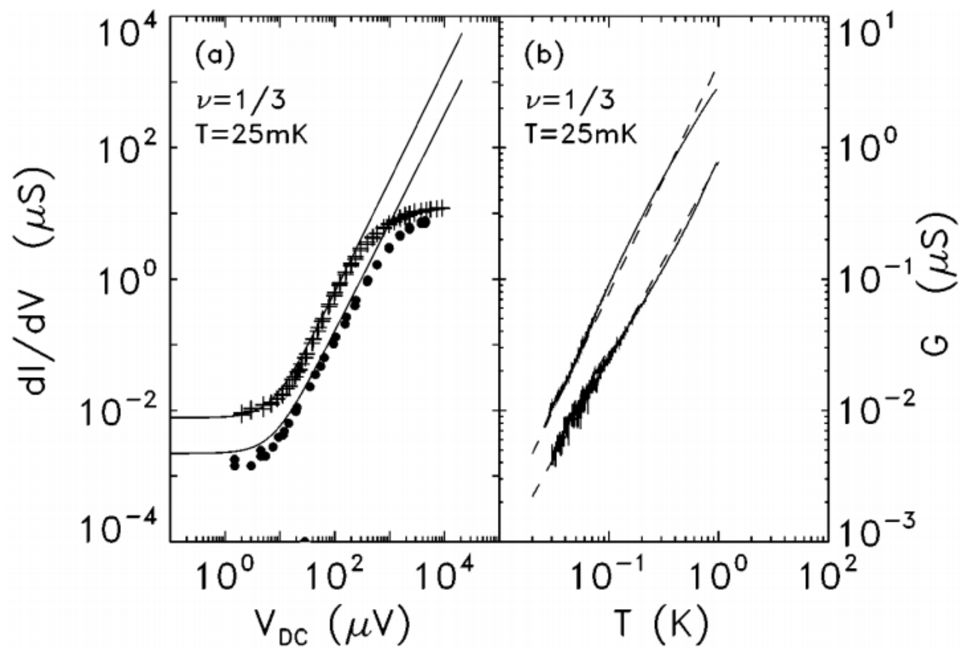
Edge States Provide a Window into the Bulk Physics

Tunneling into the edge from a metallic lead

$$I \sim V^3$$

Compared with $\upsilon = 1$

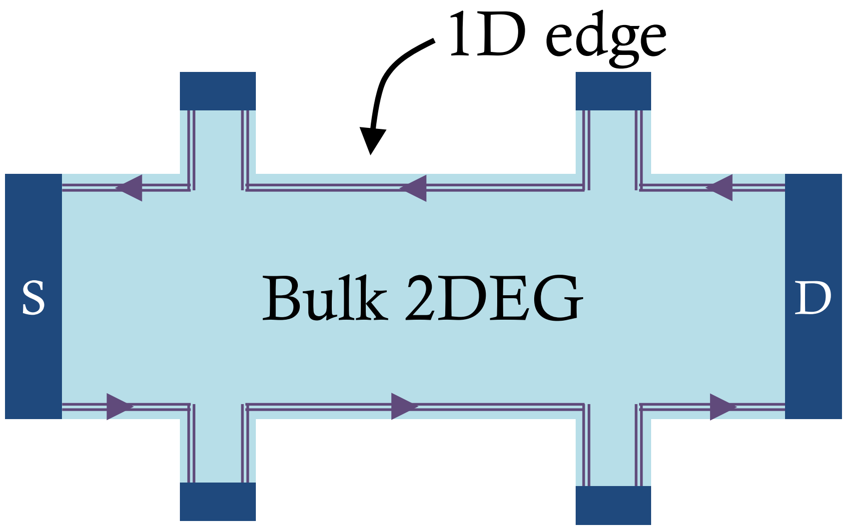
$$I \sim V$$



Exp: Chang, Pfeiffer, West

Theory: Kane-Fisher; Fendley-Ludwig-Saleur

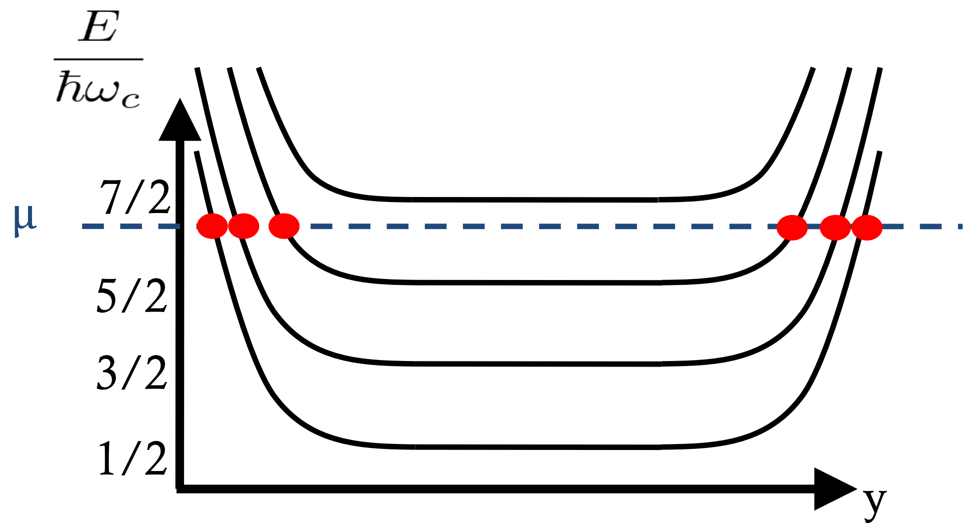
When can multiple, distinct edges bound the same bulk phase?



Experimentally relevant examples bulks with distinct edge phases include (cleanest signatures):

IQH $\nu \geq 8$, FQH $\nu = 8/3, 16/5, 16/7, \ldots$

Integer Quantum Hall Edge



$$S_{\nu=n} = \int dtdx \ \sum_{i=1}^{n} (\psi_R^{(i)})^\dagger i(\partial_t - v_i \partial_x)\psi_R^{(i)}$$

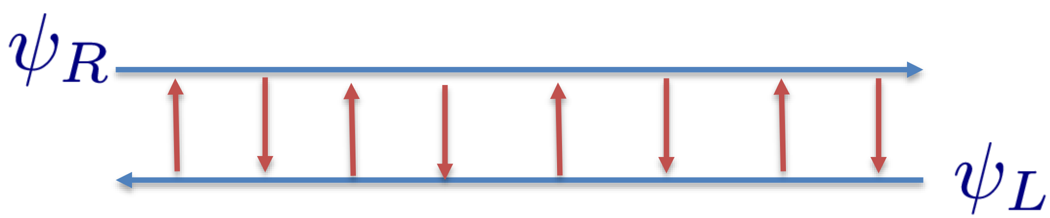
$$\psi_R^{(i)}(x)\psi_R^{(j)}(y) = (-1)\psi_R^{(j)}(y)\psi_R^{(i)}(x)$$

Non-Chiral Integer Edge

$$S_{\nu=1+\nu=-1} = \int dtdx \left[\psi_R^\dagger i(\partial_t - v_R\partial_x)\psi_R + \psi_L^\dagger i(\partial_t + v_L\partial_x)\psi_L\right]$$

Chiral edges are stable,
non-chiral edges are generically unstable.

$$\delta S_{\nu=1+\nu=-1} = M \int dtdx \left[\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R\right]$$



Fractional Quantum Hall Edges

$$S_{\nu=1/k} = \int dtdx\,\left[\psi_R^\dagger i(\partial_t - v_R\partial_x)\psi_R + f(k)\lim_{\epsilon\to 0}(\psi_R^\dagger\psi_R)(x)(\psi_R\psi_R)(x+\epsilon)\right]$$

$$S_{\nu=1/k} = \frac{k}{4\pi}\int dtdx\,\left[\partial_x\phi(\partial_t - v_R\partial_x)\phi\right]$$

$$f(1) = 0, \quad f(k\neq 1) > 0$$

Bosonization:

$$\psi_R \leftrightarrow e^{ik\phi}$$

$$\psi_R^\dagger i(\partial_t - v_R\partial_x)\psi_R \leftrightarrow \frac{1}{4\pi}\partial_x\phi(\partial_t - \partial_x)\phi$$

$$\partial_t(\partial_x\phi) - \partial_x(v_R\partial_x\phi) = 0$$

$$\lim_{\epsilon\to 0}(\psi^\dagger\psi)(x)(\psi^\dagger\psi)(x+\epsilon) \Leftrightarrow 0$$

$$\lim_{\epsilon\to 0}(\psi^\dagger\psi)(x)(\psi^\dagger\psi)(x+a) \Leftrightarrow \frac{(k-1)}{4\pi}\partial_x\phi(\partial_t - \partial_x)\phi$$

$$\nu = 1/k$$

Fractional Quantum Hall Edge

$$S_{\nu=1/k} = \frac{k}{4\pi}\int dt dx \left[\partial_x\phi(\partial_t - v_R\partial_x)\phi\right]$$

$$\phi \equiv \phi + 2\pi$$

$k = 1$

$$e^{im\phi} \quad m \in \mathbf{Z}$$

$$\langle\psi^\dagger(x)\psi(0)\rangle = \frac{1}{x - vt}$$

general k

$$\cancel{e^{2.13534 i\phi}}$$

$$\langle e^{im\phi(x)} e^{-im\phi(0)}\rangle = \frac{1}{(x - v_R t)^{\frac{m^2}{2k}}}$$

vs.

GOOD

$$\psi(x)\psi(y) = (-1)\psi(y)\psi(x)$$

BAD – NOT ALLOWED

$$e^{im\phi(x)}e^{in\phi(y)} = e^{\pi i\frac{mn}{k}}e^{in\phi(y)}e^{im\phi(x)}$$

Multiple Edge Modes

$$S = \frac{1}{4\pi}\int dtdx\,\left[\boxed{K_{IJ}}\partial_x\phi_I\partial_t\phi_J - \boxed{V_{IJ}}\partial_x\phi_I\partial_x\phi_J + \boxed{2t_I\epsilon_{\mu}}{}_{\nu}\partial_\nu\phi_I A_\mu\right]$$

right-moving modes - # left-moving modes = signature of K_{IJ}

$\nu = 0$

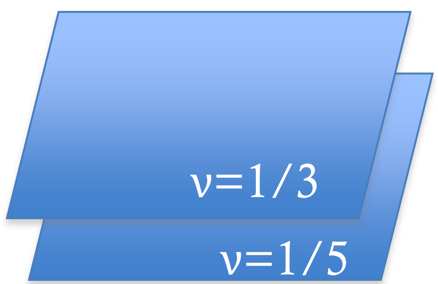
$\nu = t_I (K^{-1})^{IJ} t_J$

Wen *Int. J. Mod. Phys.* **B6** 1711 (1991)

Examples of K-matrices

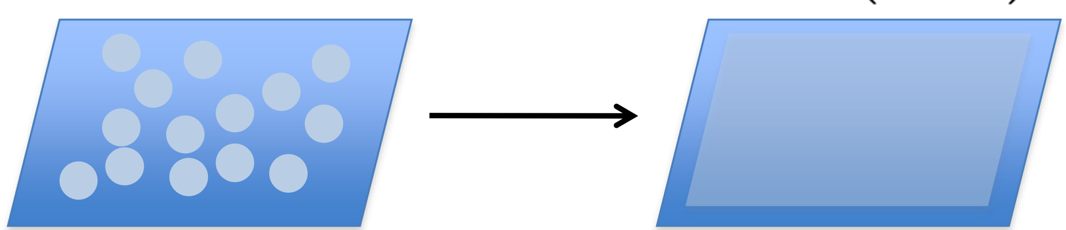
IQH: $\nu = N$ $\quad K = \mathbb{I}_N$

Laughlin: $\nu = 1/m$ $\quad K = (m)$



Bilayer system: $\nu = 1/3 + 1/5$ $\quad K = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

Hierarchy: $\nu = 2/5$ $\quad K = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$



Bulk-Edge Correspondence

$$S = \frac{1}{4\pi}\int dtdx\,\left[K_{IJ}\partial_x\phi_I\partial_t\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J + 2t_I\epsilon_{\mu\nu}\partial_\nu\phi_I A_\mu\right]$$

$$\text{parity}(K_{IJ})$$

$$\kappa_{xy} \sim \text{signature}(K_{IJ})$$

$$\sigma_{xy} = t_I(K^{-1})^{IJ}t_J\frac{e^2}{h}$$

$$\text{quasiparticles} \leftrightarrow m_I$$

$$\text{charge}(m_I) = m_I(K^{-1})^{IJ}t_J$$

$$\text{statistics}(m_I, n_J) = \exp(2\pi i m_I(K^{-1})^{IJ}n_J)$$

$$S_{CS} = \int dtdxdy\;\epsilon_{\mu\nu\rho}\Big[\frac{K_{IJ}}{4\pi}a_\mu\partial_\nu a_\rho + 2t_I A_\mu\partial_\nu a_\rho\Big]$$

Redundancy of Edge (and Bulk) Descriptions

Are these two theories the same (ignoring the charge vector)?

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad K_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Yes. $$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Same operators and scaling dimensions;
preserves excitation spectrum

$\Updownarrow$

$$K_2 = W^{tr} K_1 W, \;\; W \in GL(n, \mathbf{Z})$$

Read, *PRL* **65** 1502 (1990); Fröhlich and Thiran *J. Stat. Phys.* **76**, 209 (1994)

Distinct classes of K-matrices can (almost always) be distinguished by their scaling dimensions

Plamadeala, Nayak, MM

If K is chiral, scaling dimensions are universal:

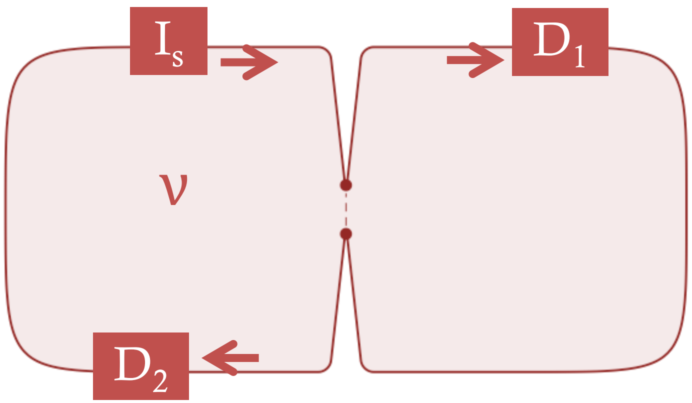
$$\langle e^{im_I \phi^I(0,t)} e^{-im_J \phi^J(0,0)} \rangle \sim \frac{1}{t^{\Delta_m}}$$

Scaling dimension: $\Delta_m = \frac{1}{2} m_I K^{-1}_{IJ} m_J$

When K is non-chiral, Δ depends on V-matrix

Scaling dimensions can be used to physically distinguish edge phases

1) Tunneling across a QPC



$$\mathcal{L}_{tun} = \sum_{m_I} v_m e^{i m_I \phi_R^I} e^{-i m_I \phi_L^I} + h.c.$$

Creates R-mover

Annihilates L-mover

Many terms: most relevant minimizes $mK^{-1}m$

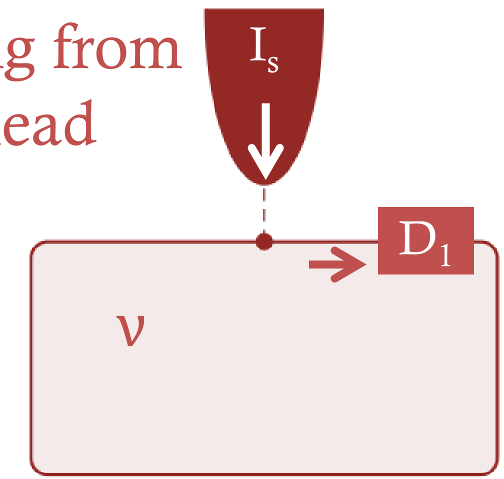
Expect most relevant term to dominate backscattered current:

$$I_b \propto |v_m|^2 V^{2mK^{-1}m-1}$$

Chamon, Freed Wen (1994)
Kane and Fisher (1992)

Scaling dimensions can be used to physically distinguish edge phases

2) Tunneling from a metallic lead



Tunnel one electron:

$$\mathcal{L}_{lead} = t_m \psi^\dagger_{lead} e^{i m_I \phi^I_R}$$

$$mK^{-1}t = 1$$

Tunnel two electrons:

$$\mathcal{L}_{lead} = t_m \psi^\dagger_{lead} \partial \psi^\dagger_{lead} e^{i m_I \phi^I_R}$$

$$mK^{-1}t = 2$$

Most relevant term minimizes $mK^{-1}m + n^2$

$$I_{lead} \propto |t_m|^2 V^{mK^{-1}m + n^2 - 1}$$

Tunnel n electrons:

$$\mathcal{L}_{lead} = t_m \underbrace{\left[\psi^\dagger_l \partial \psi^\dagger_l \partial^2 \psi^\dagger_l \ldots \right]}_{n \text{ terms}} e^{i m_I \phi^I_R}$$

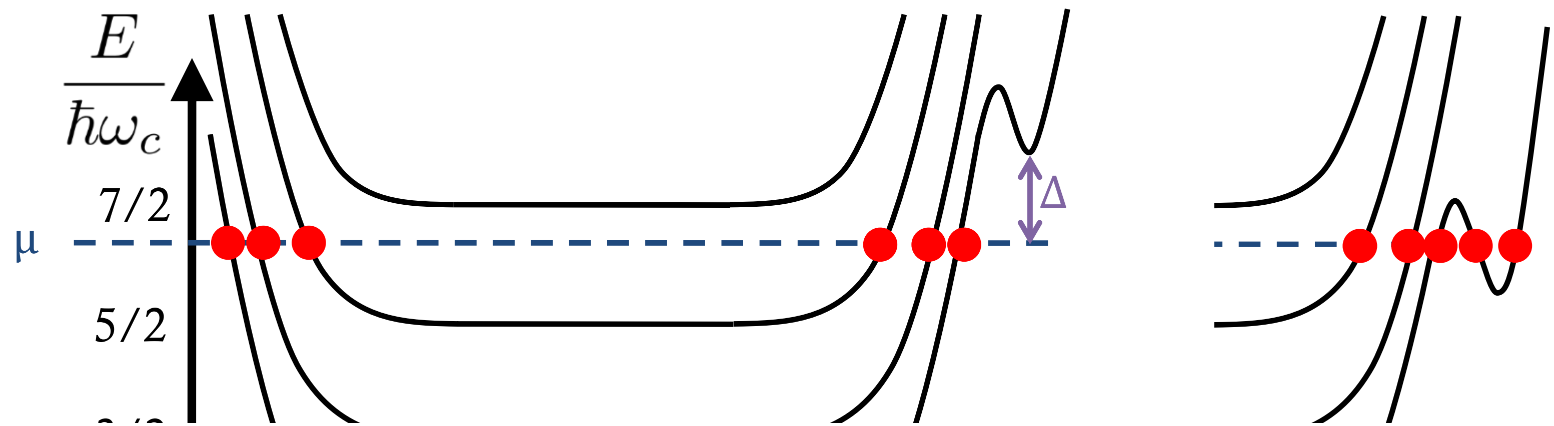
$$mK^{-1}t = n$$

Chamon, Freed Wen (1994)   Kane and Fisher (1992)

EDGE PHASE TRANSITIONS

Plamadeala, MM, & Nayak
Cano, Cheng, MM, Nayak, Plamadeala, &Yard

The confining potential matters



Can interactions with gapped edge modes change the phase of the edge?

Chamon & Wen

Append the New Modes to the Existing K-matrix

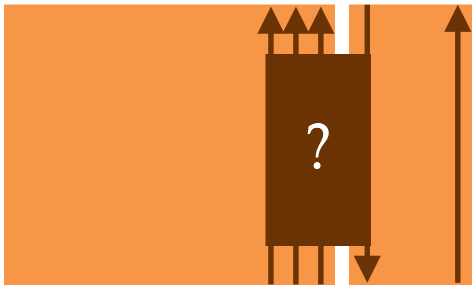
$$K \to K \oplus \sigma_z$$
$$t \to (t, 1, 1)$$

$$\begin{pmatrix} 1 & 2 & 3 & \cdots \\ 2 & 5 & 7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \to \left(\begin{array}{cccc|cc} 1 & 2 & 3 & \cdots & 0 & 0 \\ 2 & 5 & 7 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{array}\right)$$

ν= 1 strip

(New quasiparticles $e^{i\phi_{N+1}}, e^{i\phi_{N+2}}$ are electrons)

When could inter-edge tunneling open a gap?



Given an inter-edge Tunneling Operator:

$$e^{in_I\phi_I} + h.c. \propto \cos(n\phi)$$

Requirements on the Operator

Conserves Electrical Charge $\quad n_I K^{-1}_{IJ} t_J = 0$

Spin-0: $\quad n_I K^{-1}_{IJ} n_J = 0$

Not met for a chiral edge. Example:

$$K = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad n_I (K^{-1})^{IJ} n_J = \frac{n_1^2}{3} + n_2^2 - n_3^2 = 0$$

$$\Longleftrightarrow \quad n_1 = 0$$

Haldane *PRL* **74** 2090 (1995)

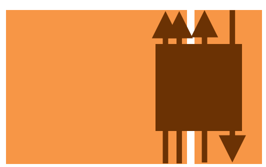
Edge Transitions: Example 1

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}, t_1 = (1, -1)^T$$

Enlarge: $K_1 \to K_1 \oplus \sigma_z \qquad t_1 \to (1, -1, 1, 1)^T$

1) 

$$S' = \int dx dt u' \cos(\phi_3 + \phi_4)$$

2) 

$$S'' = \int dx dt u'' \cos(\phi_1 - 11\phi_2 + 2\phi_3 + 4\phi_4)$$

$$n = (1, -11, 2, 4)$$

Strategic variable change $\phi = W\phi'$

$$\begin{cases} S'' = \int dx dt u'' \cos(\phi'_3 + \phi'_4) \\ K \to K_2 \oplus \sigma_z \end{cases}$$

$$K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$$

Is the resulting theory the same or different?

Example, cont

Different!

1) Most relevant term:

$$u' \cos(\phi_3 + \phi_4)$$

2) Most relevant term:

$$u'' \cos(\phi_3' + \phi_4')$$

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}, t_1 = (1, -1)^T$$

$$K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, t_2 = (-1, -2)^T$$

Are these phases distinct? $\Delta = \frac{1}{2} m_I K_{IJ}^{-1} m_J$

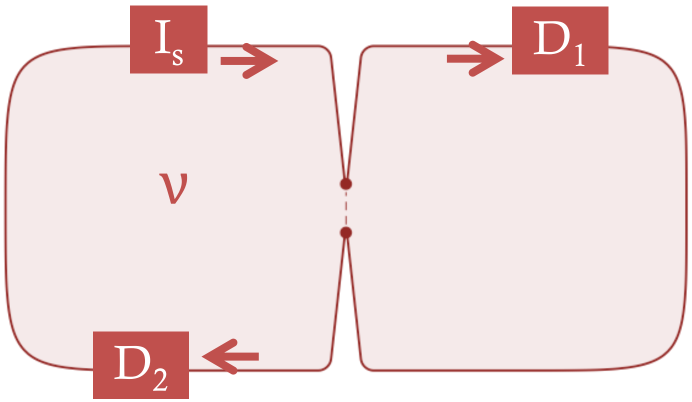
$$\Delta_{min} = 1/11$$

$$\Delta_{min} = 3/11$$

Tunneling Distinguishes the Edges

Example, cont

1) Tunneling across a QPC

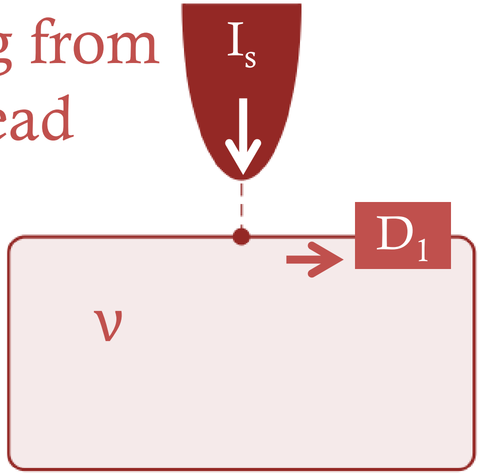


$$I_b \propto V^{2\Delta_{min}-1}$$

$$K_1 \rightarrow \quad I_b^{(1)} \propto V^{-9/11}$$

$$K_2 \rightarrow \quad I_b^{(2)} \propto V^{-5/11}$$

2) Tunneling from a metallic lead



Both edges have a charge *e* operator, but different scaling dimensions

$$I_{lead} \propto |t_m|^2 V^{mK^{-1}m}$$

$$K_1 \rightarrow I_{lead}^{(1)} \propto V \qquad K_2 \rightarrow I_{lead}^{(2)} \propto V^3$$

Edge Transitions: Example 2
Bose-Fermi Transitions

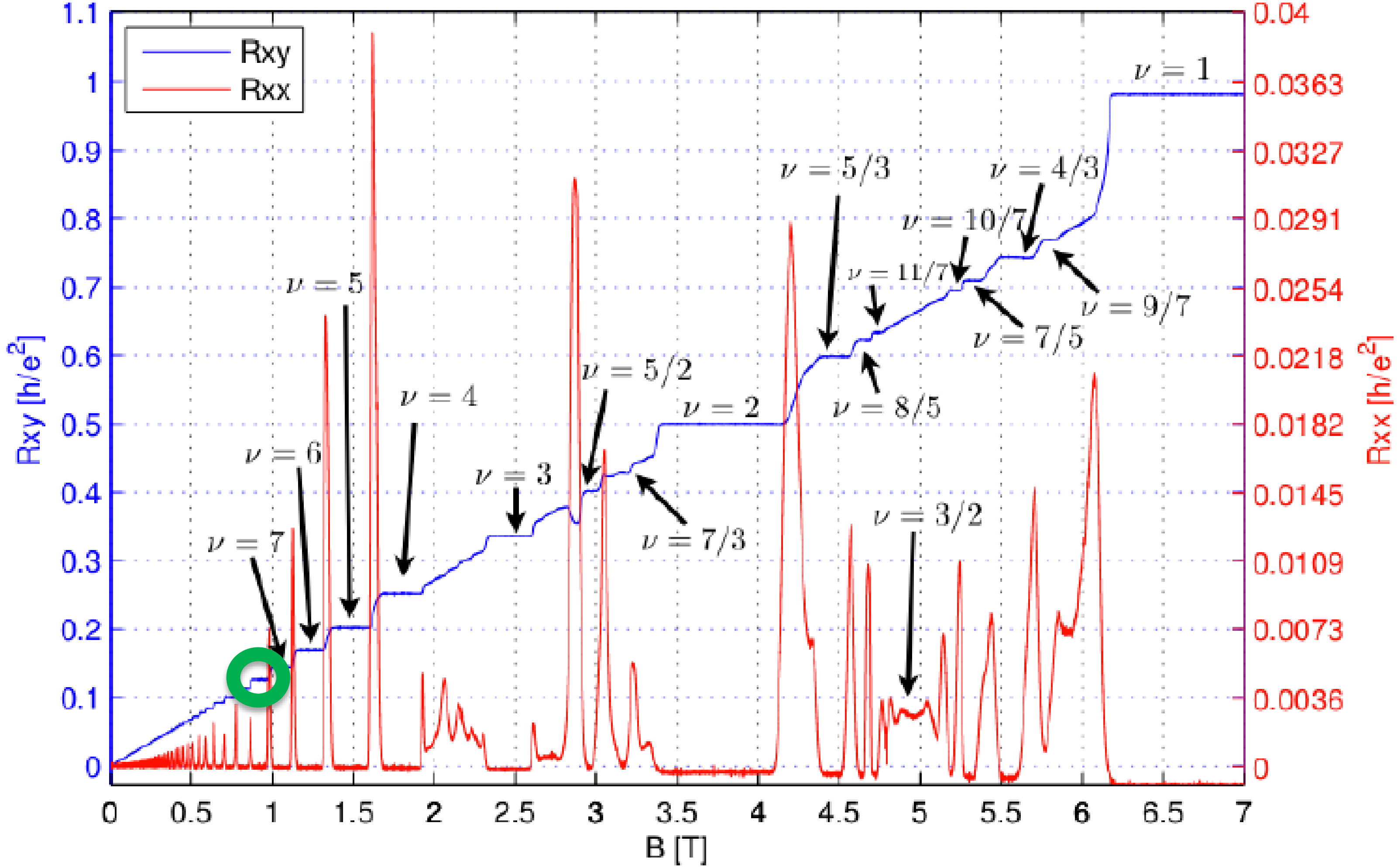
$$K_{\mathrm{odd}} \oplus \sigma_z = W^T (K_{\mathrm{even}} \oplus \sigma_z) W$$

Example

IQH ν = 8: $K = \mathbb{I}_8$ $\quad W_8^T \left(K_{E_8} \oplus \sigma_z \right) W_8 = \mathbb{I}_8 \oplus \sigma_z$

$$\mathbb{I}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

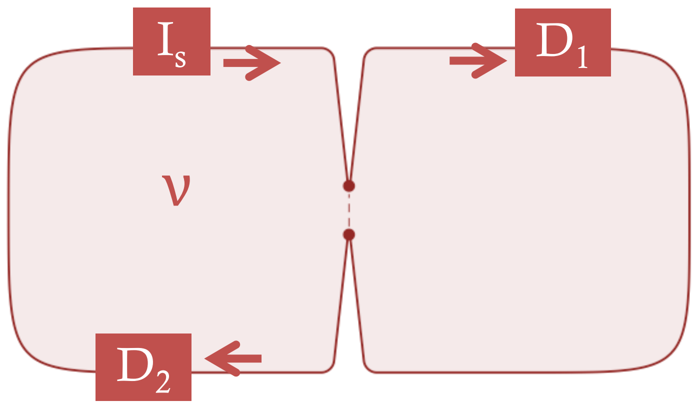
Even lattice!



Rxy
Rxx
Rxy [h/e2]
Rxx [h/e2]
B [T]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
0
0.0036
0.0073
0.0109
0.0145
0.0182
0.0218
0.0254
0.0291
0.0327
0.0363
0.04
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
ν = 1
ν = 5/3
ν = 4/3
ν = 10/7
ν = 11/7
ν = 9/7
ν = 7/5
ν = 8/5
ν = 5
ν = 4
ν = 5/2
ν = 2
ν = 6
ν = 3
ν = 7/3
ν = 3/2
ν = 7

Can distinguish experimentally between candidate ν = 8 phases

1) Tunneling across a QPC



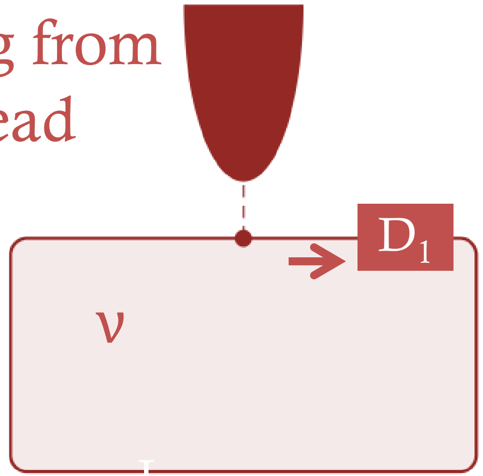
$$I_b \propto V^{2\Delta_{min}-1}$$

$\mathbb{I}_8 \to I_b \propto V$ (electron tunneling)

$K_{E_8} \to I_b \propto V^3$ (charge 2*e* tunneling of composite particle)

Can distinguish experimentally between candidate ν = 8 phases

2) Tunneling from a metallic lead



In I_8 state tunnel one electron:

$$\mathcal{L}_{lead} = t_m \psi^\dagger_{lead} e^{i\phi_I}$$

$$I_{lead} \propto V$$

E_8 state does not have a charge *e* operator!

➡ Tunnel two electrons from lead:

Spin-polarized: $\mathcal{L}_{lead} = t_m \psi^\dagger_{lead} \partial \psi^\dagger_{lead} e^{im_I \phi^I_R}$ $\quad I_{lead} \propto V^5$

Not spin-polarized: $\mathcal{L}_{lead} = t_m \psi^\dagger_{lead,\uparrow} \psi^\dagger_{lead,\downarrow} e^{im_I \phi^I_R}$ $\quad I_{lead} \propto V^3$

Physical Criteria for Distinct Edges

When do distinct edge phases exist for a given bulk?

Answer:

- Two distinct edges (different tunneling exponents) can border the same bulk when the braiding matrix for the bulk quasiparticles inferred from the edge modes is the same.
- The edge transition does not necessarily preserve the total number of edge modes.
- It may be necessary to add additional $\upsilon = 1$ modes.

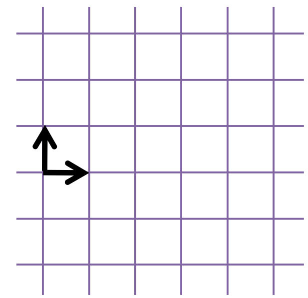
MATHEMATICAL FORMULATION

Equivalence class of K-matrices = lattice

K-matrix

Lattice $\Lambda = \{m_I \mathbf{e}_I | m_I \in \mathbb{Z}\}$

$K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$



Example: K = (3) $\Lambda = \sqrt{3}\mathbf{Z}$

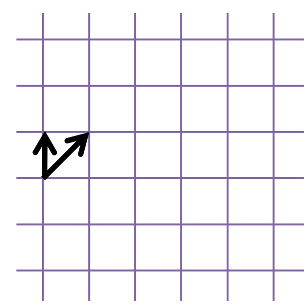
Edge phase = lattice
= K-matrix equivalence class

Choose new basis:

$\Lambda = \{m_I \mathbf{e}'_I | m_I \in \mathbb{Z}\}$

$K'_{IJ} = \mathbf{e}'_I \cdot \mathbf{e}'_J$

$W^T K W = K'$



Quasiparticles and Operators

$$\Lambda^* = \{m_I \mathbf{f}_I | m_I \in \mathbb{Z}\} \qquad f_a^I = \left(K^{-1}\right)^{IL} e_{La}$$

Example: K = (3), $$\Lambda^* = \frac{1}{\sqrt{3}}\mathbf{Z}$$

Inner product $\Leftrightarrow$ statistics $\Leftrightarrow$ scaling dimension (on diagonal)

$$\mathbf{f}_I \cdot \mathbf{f}_J = K^{-1}_{IJ}$$

$$S_{m,m'} \propto e^{-2\pi i m^T K^{-1} m'}$$

Want to classify a bulk phase by its "primitive" quasiparticles

Λ^* Dual lattice vectors = quasiparticles

Λ Original lattice vectors = trivial particles

Λ^*/Λ "Discriminant group" =
Group of primitive quasiparticles =
quasiparticles modulo the first particle with trivial statistics with all other particles

Example: K = (3), Discriminant group = $\mathbf{Z}_3$

$$\begin{tikzcd} \text{Lattices} \arrow[r, "\text{many-to-one}"] \arrow[d, leftrightarrow] & \text{Discriminant groups} \\ \text{Edge phases} \arrow[r, "\text{many-to-one}"] & \text{Bulk phase} \arrow[u] \end{tikzcd}$$

Stable Equivalence

Edges Are Stably Equivalent if …

$$\Lambda_1^*/\Lambda = \Lambda_2^*/\Lambda$$

$$f_I^{(1)} \cdot f_J^{(1)} = f_I^{(2)} \cdot f_J^{(2)} + \mathrm{mod}\ 2$$

Chiral transitions if:

$$\mathrm{signature}(\Lambda_1) = \mathrm{signature}(\Lambda_2)$$

Nikulin *Math. USSR Izv.* **14**, 103 (1980)

Formalize the odd-even correspondence

Utilize another theorem from Nikulin:

For every fermionic bulk phase, there is a corresponding* edge phase which yields the same** bulk quasiparticles and statistics and has no gapless charge e operator

Caveats
*might have different number of edge modes
**mod e

Nikulin *Math. USSR Izv.* **14**, 103 (1980)

Future Directions

- Generalizations – non-Abelian, higher-dimensions, interplay of symmetry.

- Could there be a physical mechanism, e.g., disorder, that initiates the flow of V?

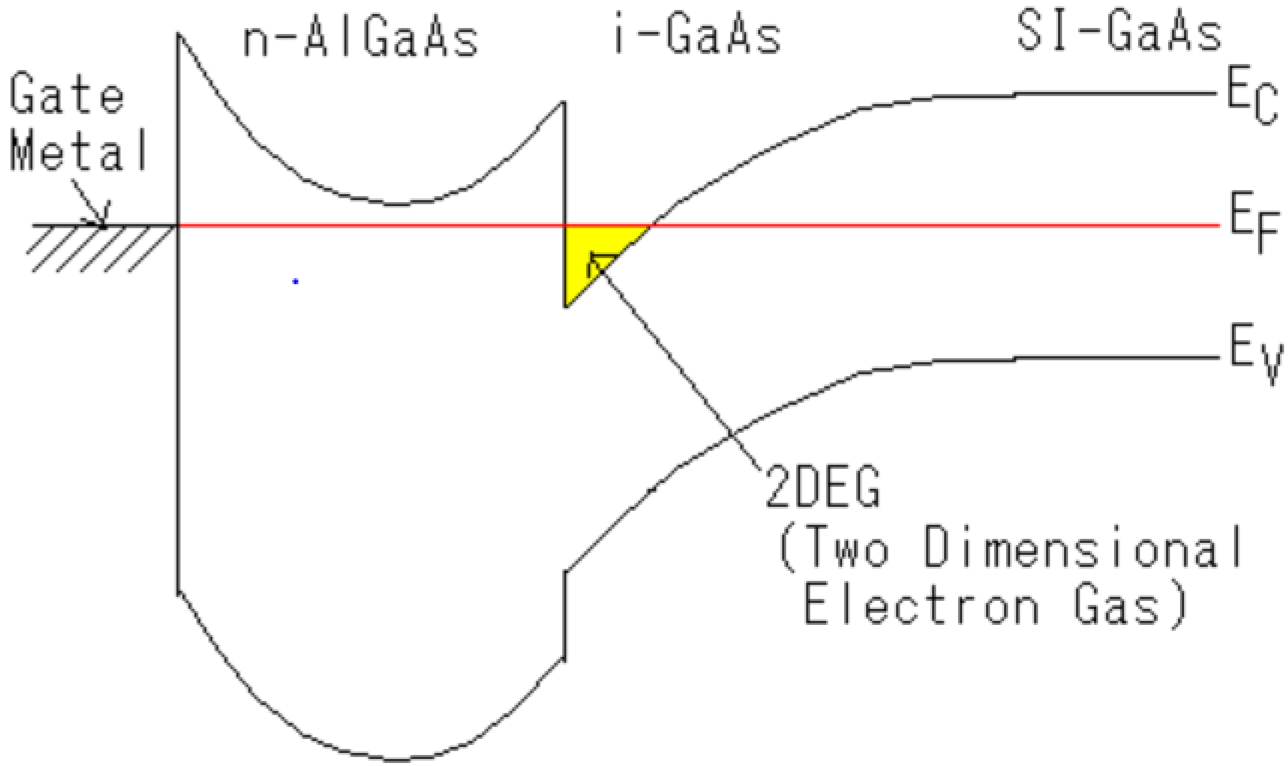
- Many more edge transitions between not-fully-chiral edge states, e.g., 1/3 state.

- There exists an intriguing analogy between 4-manifold topology and the Abelian Hall states. Does this analogy run deeper?

Conclusions

- A bulk chiral quantum Hall phase generically has multiple edge phases.
- These phases can be distinguished experimentally using tunneling measurements.
- Every fermionic edge phase has a corresponding bosonic edge phase

Heterostructure Band Energy Diagram



More topologically trivial additions

Superfluid strip

$$K \to K \oplus \sigma_x$$

Several ν= 1 strips

$$K \to K \oplus \sigma_z \oplus \sigma_z \oplus \cdots \oplus \sigma_z$$

Strip of anything non-chiral with trivial quasiparticles

$$K \to K \oplus L, \quad |\det(L)| = 1 \quad \mathrm{sig}(L) = (n, n)$$

In principle, can count the lattices in a genus by the Smith-Siegel-Minkowski mass formula

Sum over lattices in a genus = sum over edge phases

$$\sum_{\Lambda \in g} \frac{1}{|\mathrm{Aut}(\Lambda)|} = m(K)$$

Conway and Sloane 1988

# of automorphisms of lattice counts symmetries of K matrix

- Chiral Abelian quantum Hall states with more than 10 edge modes have multiple distinct chiral edge phases[1]
- Otherwise there is a finite set of bulk states with only one edge phase; all others have multiple[2]

→Multiple edge phases are the norm, not the exception

[1]G. Watson, Proc. London Math Soc. **12** 57787 (1962)
[2]D. Lorch and M. Kirschmer, LMS Journal of Computation and Mathematics **16**, 172 (2013)

Example: even-odd equivalence by adding edge modes

$$\nu = 1/5$$
$$K = (5)$$

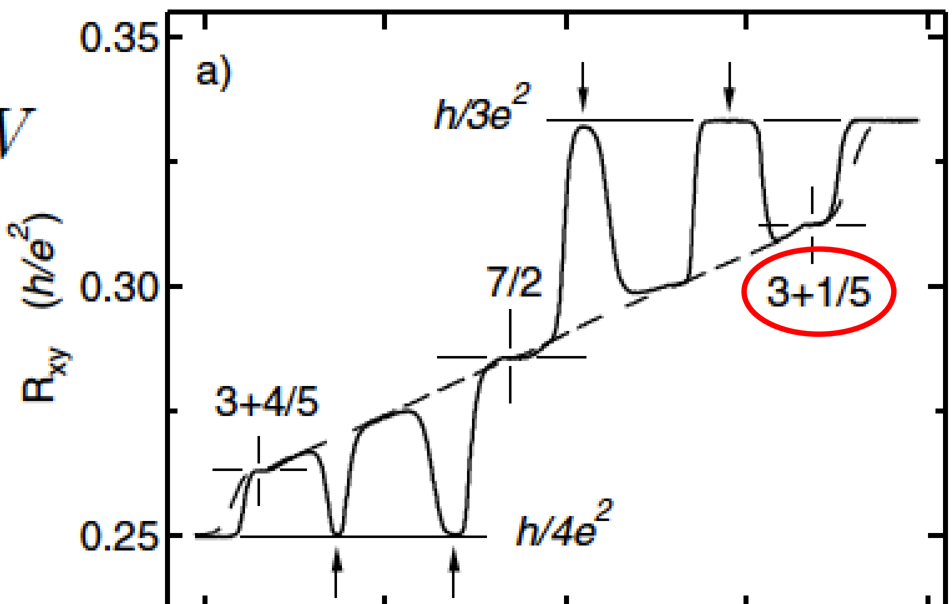
No non-trivial stable equivalence preserving full chirality

$$\nu = 3 + 1/5$$
$$K_1 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$K_1 \oplus \sigma_z = W^T (K_2 \oplus \sigma_z) W$$

$$K_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Even lattice

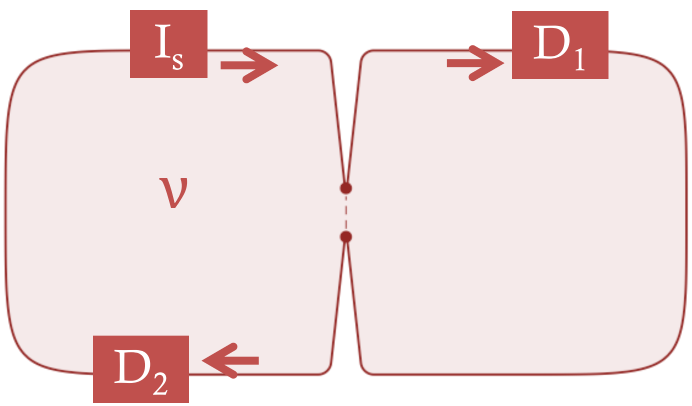


Eisenstein, et al, PRL **88**, 076801 (2002)

Candidate states at ν=3+1/5 distinguishable by experiment

Example, cont

1) Tunneling across a QPC

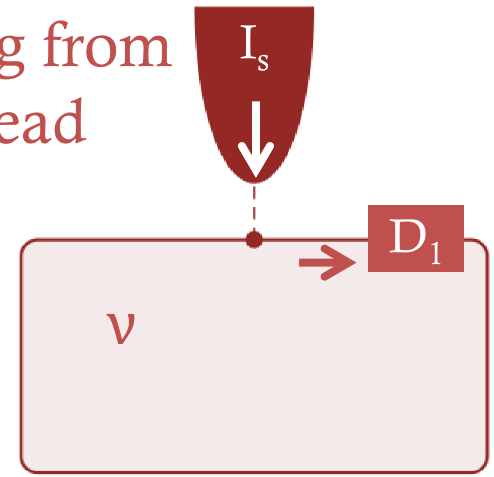


$$I_b \propto V^{2\Delta_{min}-1}$$

$$K_1 \rightarrow \quad I_b^{(1)} \propto V^{-3/5}$$

$$K_2 \rightarrow \quad I_b^{(2)} \propto V^{3/5}$$

2) Tunneling from a metallic lead



New edge does not have electron

$K_1 \rightarrow I_{lead}^{(1)} \propto V$ (tunnel electron)

$K_2 \rightarrow I_{lead}^{(2)} \propto V^5$ (tunnel 2 electrons, spin polarized)

Candidate states at ν=3+1/5 have same quasiparticles mod *e*

$$K_1 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$t = (1, 1, 1, 1)$$

$$K_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$t = (2, 2, 0, 0)$$

Compare quasiparticle charge:

$$e^*/e \in \{0, 1/5, 2/5, 3/5, 4/5\}$$

Defined mod *e*

$$e^*/e \in \{0, 2/5, 4/5, 6/5, 8/5\}$$

Defined mod 2*e*