

Fermion space charge in narrow band-gap semiconductors, Weyl semimetals and around highly charged nuclei

Eugene B. Kolomeisky
University of Virginia

Work done in collaboration with:

- J. P. Straley, University of Kentucky
- H. Zaidi, University of Virginia
- Some is published in [Phys. Rev. B 88, 165428 \(2013\)](#)
- Supported by US AFOSR



- QED is the most successful and best verified physics theory.
- Observables are computed via perturbation theory in $\alpha = e^2/\hbar c = 1/137$.
- Measurements impressively match calculations.
- Is there any new physics left to understand and observe?
- Yes, strong field effects!
- Where to look for them?
- Bound states of nuclei of large charge Ze !
- In calculations involving bound states α also appears in the $Z\alpha$ combination.
- Even though $\alpha \ll 1$, $Z\alpha$ may not be...
- What happens if $Z\alpha > 1$ ($Z > 137$)? Non-perturbative effects!
- Pomeranchuk&Smorodinsky (1945), Gershtein&Zel'dovich (1969): vacuum becomes unstable with respect to creation of electron-positron pairs; positrons leave physical picture while the fermion space charge ("vacuum" electrons) remains near the nucleus screening its charge.
- Greiner et al. (1969), Popov (1970): vacuum condensation begins at a critical charge Z close to 170.
- But $Z > 170$ nuclei are not available! How to observe the effect?
- Slowly colliding U nuclei have combined $Z = 184$ exceeding 170!
- Experiments (1978-1999, GSI, Germany) failed.

These issues are still worth pursuing...

- Because these kinds of problems have condensed matter counterparts: **impurity states in semiconductors**.
- *Materials are available*: narrow-band gap semiconductors (*InSb* type) and Weyl semimetals (very recently, **2014**, observed in *Na₃Bi* and *Cd₃As₂*).
- Material parameters are such as **critical charge is modest and readily achievable**.

Outline

- Critical charge problem in QED and condensed matter physics.
- Supercritical regime: Thomas-Fermi theory and its solution - prediction of nearly-universal observable charge.
- Conclusions.

Critical charge in QED: heuristic argument for the Dirac-Kepler problem

- What is the ground-state energy of an electron in the field of charge Ze ?

- Classical energy:
$$\varepsilon = c\sqrt{p^2 + m_e^2 c^2} - \frac{Ze^2}{r}$$

- The uncertainty principle:
$$r \geq \hbar/p$$

- Combine:
$$\varepsilon(p) \gtrsim c \left(\sqrt{p^2 + m_e^2 c^2} - zp \right), \quad z = Z\alpha$$

- Minimize with respect to free parameter p :

$$p_0 \simeq \frac{m_e c z}{\sqrt{1-z^2}}, \quad r_0 \simeq \frac{\hbar}{p_0} \simeq \lambda \frac{\sqrt{1-z^2}}{z}, \quad \lambda = \frac{\hbar}{m_e c} = \frac{r_e}{\alpha}$$

- The lowest (ground-state) energy:
$$\varepsilon_0 = m_e c^2 \sqrt{1-z^2}$$

Consequences

- $z \ll 1$ - non-relativistic H-like ion of size $\lambda/z = a_B/Z$.
- $z \rightarrow 1-0$ - ground-state sharply localized, the ground-state energy vanishes.
- Analysis becomes meaningless for $z > z_c = 1$ - mass independent?
- What about the Weyl-Kepler (massless electron) problem?

Charge measured in natural units of $1/\alpha$

Classical electron radius

Compton wavelength

Bohr radius

Critical charge in QED... continued...

- The problem is fully characterized by the Compton wavelength λ and dimensionless charge z . Dimensional analysis dictates that if there is a critical z , it cannot depend on λ , thus implying mass-independence of $z_c \simeq 1$.
- The $z > 1$ anomaly persists in the Weyl-Kepler problem: $\varepsilon'(p) \simeq pc(1 - z)$
- $z < 1$ - particle delocalized, no bound states.
- $z > 1$ - sharp localization, infinitely negative ground-state energy.

What does it all mean?

*This is a strong field limit of the **Schwinger effect**: creation of electron-positron pairs in vacuum in a uniform electric field: the work of the field to separate the constituents of the pair over Compton wavelength equals the rest energy of the pair, $eE_S\lambda \simeq m_e c^2$, or*

$$E_S = \frac{m_e^2 c^3}{e\hbar}$$

- $E \lesssim E_S$ - pairs created by tunneling; vacuum is in a metastable state.
- $E \gtrsim E_S$ - pairs created spontaneously; vacuum is absolutely unstable.
- For the Coulomb problem the instability sets in when $Ze/\lambda^2 \simeq E_S$ which again predicts $z_c \simeq 1$.

Connection to quantum-mechanical “fall to the center” effect

- Conservation of energy for classical non-relativistic electron of energy \mathcal{E} and angular momentum M moving in a central field $U(r)$ determines the range of motion:

$$p_r^2 = 2m_e\mathcal{E} - 2m_eU(r) - \frac{M^2}{r^2} > 0, \quad p^2 = p_r^2 + M^2/r^2$$

- The particle reaches the origin (falls to the center) if $\lim_{r \rightarrow 0} (r^2 U(r)) < -\frac{M^2}{2m_e}$.
- For $M=0$ the fall occurs for potential more attractive than $-1/r^2$.
- “Introduce” quantum mechanics via Langer substitution: $M^2 \rightarrow \hbar^2(l + 1/2)^2$.
- Smallest $M^2 = \hbar^2/4$ (zero-point motion).
- The fall occurs for potentials more attractive than $U_c(r \rightarrow 0) = -\frac{\hbar^2}{8m_e r^2}$.
- If this is the case, there is no lower bound on the spectrum.
- Repeat the argument for relativistic particle.

“Fall to the center” of relativistic particle

- Conservation of energy: $\mathcal{E} = c\sqrt{p_r^2 + M^2/r^2 + m_e^2 c^2} + U(r)$
- Bound states: $-m_e c^2 < \mathcal{E} < m_e c^2$
- Instability with respect to pair creation: $\mathcal{E} < -m_e c^2$
- Range of motion at lower bound $\mathcal{E} = -m_e c^2$:
$$p_r^2 = \frac{1}{c^2} (U^2(r) + 2m_e c^2 U(r)) - \frac{M^2}{r^2} > 0$$
- If $U(r)$ diverges at the origin, the fall to the center occurs if $\lim_{r \rightarrow 0} (rU(r)) < -Mc$.
- Classically ($M=0$) this occurs for potential that is more attractive than the Coulomb potential.
- Quantum-mechanically ($M = \hbar/2$) the fall occurs for potentials more attractive than
$$U_c(r \rightarrow 0) = -\frac{\hbar c}{2r}$$
- Compare with the Coulomb potential $U = -Ze^2/r \rightarrow z_c = Z_c \alpha = 1/2$: correct for spinless particle but misses $1/2$ for the Dirac particle due to the electron spin.
- The Dirac case cannot be fully understood semi-classically but further insight is still possible...

“Fall to the center” of relativistic particle... continued

- Compare non-relativistic and relativistic (at $\mathcal{E} = -m_e c^2$) expressions for the range of motion:

$$p_r^2 = 2m_e \mathcal{E} - 2m_e U(r) - \frac{M^2}{r^2} > 0 \quad vs \quad p_r^2 = \frac{1}{c^2} (U^2(r) + 2m_e c^2 U(r)) - \frac{M^2}{r^2} > 0$$

- Relativistic problem is equivalent to a non-relativistic problem with zero total energy and effective potential

$$U_{eff}(r) = -\frac{U^2(r)}{2m_e c^2} - U(r) + \frac{M^2}{2m_e r^2} + \text{extra terms due to spin in the Dirac case}$$

- For the Kepler problem $U(r) = -Ze^2/r$ the particle is always **attracted** at small distances and **repelled** at large distances.

- For the Dirac-Kepler problem the role of spin terms can be (approximately) summarized in

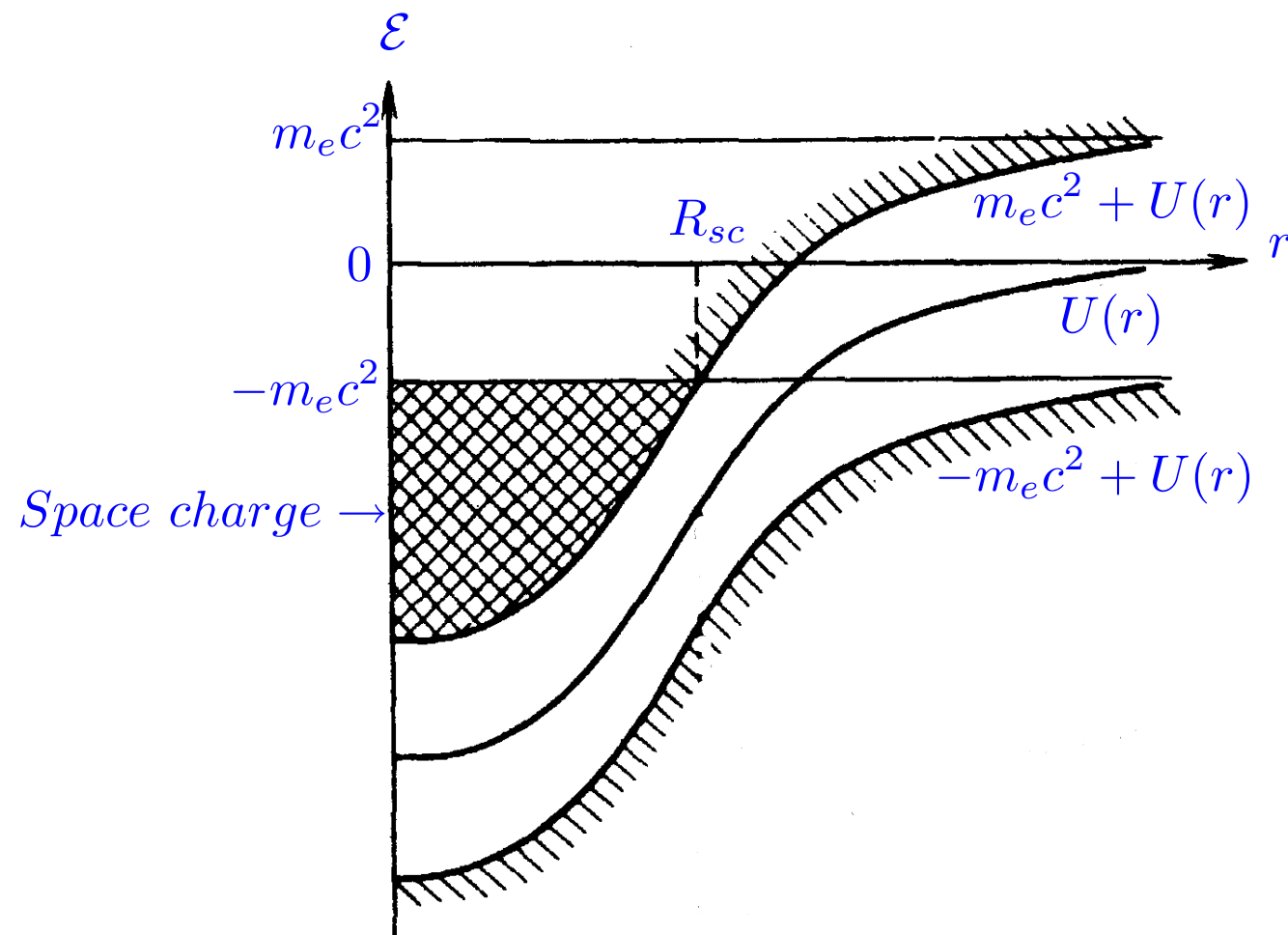
$$U_{eff}(r) = \frac{Ze^2}{r} + \frac{\hbar^2(1 - z^2)}{2m_e r^2}$$

- The fall to the center occurs for $z > 1$; the particle is confined to central region of size

$$R_{cl} = \frac{\lambda(z^2 - 1)}{2z}$$

- If nuclei were point objects, the Periodic Table would end at $Z=137$...
- Finite nuclear size: critical Z moves **up** to **170** (Greiner et al., Popov).

Semi-classical picture



- Spontaneous pair creation begins when the ground-state energy reaches the boundary of lower continuum, $\varepsilon_0 = -m_e c^2$.
- Semi-classically the size of the space charge region R_{sc} is the radius where **modified** upper continuum $m_e c^2 + U(r)$ meets **unmodified** lower continuum $-m_e c^2$, i. e. $U(R_{sc}) = -2m_e c^2$.
- This is the amount of energy needed to create a pair - positron escapes to infinity, electron remains near the nucleus.
- Accounting for finite nuclear size complicates the problem; fortunately dimensional analysis allows to anticipate the results of Popov et al.(1970+).

Critical charge in modified Dirac-Kepler problem

- *What is the critical charge of vacuum instability for finite nuclear size a ?*
- Dimensional analysis: $z_c = f\left(\frac{a}{\lambda}\right)$ or $Z_c = \frac{1}{\alpha} f\left(\frac{a}{\lambda}\right) = \frac{\hbar c}{e^2} f\left(\frac{m_e c a}{\hbar}\right)$.
- Since $z_c = 1$ for $a=0$, then $f(y \rightarrow 0) \rightarrow 1$. *For the Weyl-Kepler ($m_e = 0$) problem the critical charge is unity even in the presence of cutoff scale a !*
- As $\hbar \rightarrow 0$, Planck's constant must drop out $\rightarrow f(y \rightarrow \infty) \simeq y$ or $z_c \simeq a/\lambda$. The vacuum becomes unstable when $e\varphi(0) = 2m_e c^2$. For the uniformly charged ball model of the nucleus this translates into $f(y \rightarrow \infty) \rightarrow 4y/3$.
- The Fermi formula: $a = 0.61 r_e Z^{1/3} = 0.61 \lambda \alpha^{2/3} z^{1/3}$.
- Substitute into f : $z_c = f\left(0.61 \alpha^{2/3} z_c^{1/3}\right)$.
- *For the electron $\lambda \gg a$; the small argument limit of $f(y)$ (quantum mechanics) dominates, critical Z is slightly larger than 137 and weakly model-dependent.*
- *For the muon (200 times heavier) $\lambda \ll a$; the large argument limit of $f(y)$ (classical physics) dominates, $Z_c^{(\mu)} \simeq \left(\frac{m_\mu}{m_e}\right)^{3/2} \simeq 3000$ —model-dependent.*

Condensed matter (CM) connection: ***QED vacuum is a dielectric!***

- Excitations: electron-hole pairs (CM) \Rightarrow electron-positron pairs (QED).
- Band gap (CM) \Rightarrow combined rest energy of the electron-positron pair (QED).
- Zener tunneling in a uniform electric field (CM) \Rightarrow Schwinger effect (QED).
- Is there a CM counterpart to the $Z > 170$ vacuum instability? **Yes, moderately charged, $Z > 10$, impurity region can trigger formation of space charge.**
- Single band \Rightarrow effective mass approximation \Rightarrow non-relativistic Schrödinger equation \Rightarrow *shallow* impurity states \Rightarrow *H*-like problem of non-relativistic QM.
- **Keldysh (1963):** effective mass approximation fails to explain deep states formed near multi charged impurity centers, vacancies etc. - trapping of both electrons and holes by highly-charged recombination centers etc. These states could be associated with neither conduction nor valence band.
- **Keldysh:** two-band approximation well-obeyed in narrow-band gap semiconductors (NBGS) of the *InSb* type is needed; the *low-energy* electron-hole dispersion law is “relativistic”:

$$\epsilon(\mathbf{p}) = \pm \sqrt{(\Delta/2)^2 + v^2 p^2},$$

Diagram illustrating the relativistic dispersion law and its components:

- Conduction band** (points to the positive sign in the equation)
- Valence band** (points to the negative sign in the equation)
- Band gap** (Δ)
- “Speed of light”** (v)
- Effective mass** (m)

The equation is also written as $\Delta = 2mv^2$.

NBGS parameter values

- Electrons and holes are very “light”, $m \simeq 0.01m_e$, and “slow”, $v \approx 4.3 \times 10^{-3}c$; their band gap $\Delta \simeq 0.1eV$ is **seven** orders of magnitude smaller than the rest energy of the electron-positron pair. **Large field analog QED effects are significantly more pronounced and readily realizable!**
- The material “fine structure” constant

$$\alpha = \frac{e^2}{\hbar v \epsilon} \simeq \frac{1}{\epsilon} \simeq \frac{1}{10}$$

↑
Dielectric constant

- Keldysh:** (i) for $z = Za < 1$ the impurity states are given by the spectrum of the Dirac-Kepler problem (encompasses the theory of shallow states); (ii) $z > 1$ regime describes recombination center with “collapsed” ground state.
- Critical charge

$$z_c = f\left(\frac{a}{\Lambda}\right), \quad \Lambda = \frac{\hbar}{mv} = \frac{2\hbar v}{\Delta} = \frac{R_e}{\alpha} \simeq 10 \text{ nm}, \quad R_e = \frac{2e^2}{\epsilon\Delta} \simeq 1 \text{ nm}$$

↑
Compton wavelength

↑
Classical electron radius

- Electrons and holes are significantly more “quantum” than their QED cousins!**
No need to account for lattice structure.
- Zener’s field $E_Z = \Delta^2 / e\hbar v \simeq 10^5 V/cm$, **smaller by 10^{11}** than Schwinger’s field!

NBGS parameter values... continued

- CM “Fermi formula”, $a = 1.3R_e Z^{1/3} = 1.3\Lambda\alpha^{2/3} z^{1/3}$ which corresponds to the *external* (impurity) charge density $n_{ext} = 10^{20} cm^{-3}$ set by **1 eV** range of applicability of relativistic dispersion law.
- Critical charge equation is nearly identical to its QED counterpart:
$$z_c = f \left(1.3\alpha^{2/3} z_c^{1/3} \right)$$
- We are again in quantum mechanics dominated regime: $Z_c \simeq 1/\alpha \simeq 10$.
- Critical cluster size: $a_c \approx 3nm$.
- **$Z > 10$ impurity clusters with sizes in excess of several **nm** are more common objects than $Z > 170$ nuclei!**
- **Abrikosov&Beneslavskii (1970)**: prediction of *Weyl Semimetals* (WS); they have the gapless limit of the NBGS dispersion law, $\epsilon(\mathbf{p}) = \pm vp$. **These would realize massless version of QED where *any* field is strong!**
- In **2014** WS were discovered in Na_3Bi and Cd_3As_2 .
- In WS $Z_c = 1/\alpha \simeq 10$ *independent* of the size of the impurity region.

Summary

- At modest $Z > 10$ electrons are promoted from valence band to form a space charge around impurity cluster; the holes leave the physical picture.
- The properties of the space charge vary with Z and α and are determined by interplay of attraction to the impurity (promoting creation of space charge) and the electron-electron repulsion combined with the Pauli principle (limiting the creation of space charge).
- $Z < Z_c$: no space charge; single-particle description suffices.
- Z slightly exceeds Z_c : very few electrons are promoted to the conduction band; single -particle description is a good starting point, interactions can be accounted for perturbatively (Zel'dovich and Popov, 1971).
- $Z \gg Z_c$: the number of screening electrons is large and electron-electron interactions cannot be ignored (numerical: Greiner et al., 1975+, analytic: Migdal et al., 1976+; some interesting physics was overlooked in both of these studies).
- It will be demonstrated that ...
- The physics in the $Z \gg Z_c$ limit exhibits a large degree of universality.
- Solution to the WS problem plays a central role in understanding the NBGS/QED case.

Thomas-Fermi theory

- Physical electrostatic potential: $\varphi(\mathbf{r}) = \varphi_{ext}(\mathbf{r}) - \frac{e}{\epsilon} \int \frac{n(\mathbf{r}')dV'}{|\mathbf{r} - \mathbf{r}'|}$.
- External (impurity) potential: $\Delta\varphi_{ext} = -4\pi en_{ext}/\epsilon$, $\varphi_{ext} = \frac{Ze}{\epsilon r}$ for $r \geq a$.
- Electron number density $n(\mathbf{r})$ is only present within the region of space charge:

$$e\varphi(\mathbf{r}) > \Delta, \quad n(\mathbf{r}) > 0$$

- Radius of space charge region $R_{sc} \geq a$: $e\varphi(R_{sc}) = \Delta$, $n(R_{sc}) = 0$.

- Outside the space charge region, $n=0$, and

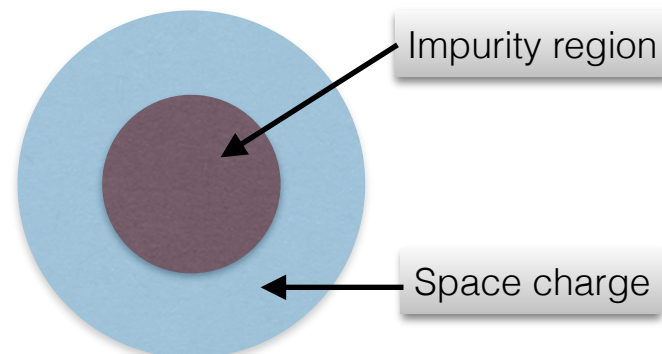
$$\varphi = \frac{Q_{\infty}e}{\epsilon r}, \quad r > R_{sc} = \frac{1}{2}Q_{\infty} \frac{2e^2}{\epsilon\Delta} \equiv \frac{1}{2}Q_{\infty}R_e$$

Observable charge
↓
Classical electron radius

- In WS case, $R_e = \infty$, the electron shell extends all the way to infinity, $R_{sc} = \infty$.
- In “natural” units of charge

$$q_{\infty} = Q_{\infty}\alpha = \frac{2R_{sc}}{\Lambda}$$

Compton wavelength



Thomas-Fermi theory... continued

- Creation of electron-hole pairs is a “chemical reaction” $e + h \rightleftharpoons 0$ with condition of equilibrium

$$\mu_e + \mu_h = 0, \mu_e = \sqrt{(\Delta/2)^2 + v^2 p_F^2} - e\varphi, \mu_h = \Delta/2$$

↑
Electron chemical potential
↑
Hole chemical potential

- The Fermi momentum: $p_F(\mathbf{r}) = \hbar \left(\frac{6\pi^2 n(\mathbf{r})}{g} \right)^{1/3}$

Degeneracy factor, ≥ 2

- Combine: $n(\mathbf{r}) = \frac{\gamma}{4\pi} \left\{ \frac{\epsilon\varphi(\mathbf{r})}{e} \frac{\epsilon}{e^2} [e\varphi(\mathbf{r}) - \Delta] \right\}^{3/2}, \quad \gamma = \frac{2g\alpha^3}{3\pi}$

Coupling constant

- $\mu_e = -\Delta$ means screening is *incomplete*; only in the WS case $\Delta = 0$ is the screening complete.

- Apply the Laplacian to $\varphi(\mathbf{r}) = \varphi_{ext}(\mathbf{r}) - \frac{e}{\epsilon} \int \frac{n(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|}$.

- Relativistic Thomas-Fermi equation (Greiner et al., Migdal et al.):

$$\nabla^2 \left(\frac{\epsilon\varphi}{e} \right) = -4\pi n_{ext} + \gamma \left\{ \frac{\epsilon\varphi}{e} \frac{\epsilon}{e^2} (e\varphi - \Delta) \right\}^{3/2}$$

Impurity region

- For $n_{ext} = 10^{20} \text{ cm}^{-3}$, a **10-nm** impurity region contains $Z \approx 400$ bare charge which is significantly larger than the critical charge of **10**.
- To first approximation mesoscopic impurity region may be viewed as charged half-space:

$$\frac{d^2}{dx^2} \left(\frac{\epsilon \varphi}{e} \right) = -4\pi n_{ext}(x) + \gamma \left\{ \frac{\epsilon \varphi}{e} \frac{\epsilon}{e^2} (e\varphi - \Delta) \right\}^{3/2}, n_{ext}(x < 0) = 3Z/4\pi a^3, n_{ext}(x > 0) = 0$$

- Deeply inside the source region local neutrality $n = n_{ext}$ holds:

$$\varphi(x \rightarrow -\infty) \equiv \varphi_{-\infty} \approx \frac{e}{\epsilon} \left(\frac{4\pi n_{ext}}{\gamma} \right)^{1/3}$$

- The source boundary is a perturbation to constant density and potential; assume the effect is weak, $\varphi = \varphi_{-\infty}(1 - \phi)$, $\phi \ll 1$, and linearize about $\varphi = \varphi_{-\infty}$ (this can be justified):

$$\frac{d^2 \phi}{dx^2} - \kappa^2 \phi = 0, \quad \kappa^{-1} \simeq (\gamma Z^2)^{-1/6} a \quad \leftarrow \text{Debye screening length}$$

- γZ^2 parameterizes degree of screening: concept of the screening length applied to finite-sized object is only applicable in the regime of strong screening, i. e. when $\gamma Z^2 \gg 1$.
- Local neutrality is only violated near the $x = 0$ boundary within the screening length. Net charge: $Q(r \leq a) \simeq \kappa^{-1} a^2 (Z/a^3) \simeq Z(\gamma Z^2)^{-1/6}$.

Regimes of screening in practical terms

- The cross-over charge $\gamma Z^2 \simeq 1$: $Z_x \simeq \gamma^{-1/2}$.
- Condensed matter and QED coupling constants γ are vastly different: $\gamma \simeq 10^{-3}$ (CM) versus $\gamma \simeq 10^{-7}$ (QED).
- Then $Z_x \simeq 30$ (CM) versus $Z_x \simeq 3000$ (QED).
- In condensed matter setting both the regimes of weak, $10 \lesssim Z \lesssim 30$, and strong screening, $Z \gtrsim 30$, are experimentally accessible.
- In QED setting studying space charge around highly charged nuclei is completely academic.

Back to the analysis of the regime of strong screening...

Outside of the impurity region; spherically-symmetric charge distribution: Weyl semimetal

- Outside of the source region we need to solve the full nonlinear equation

$$\nabla^2 \left(\frac{\epsilon\varphi}{e} \right) = \gamma \left(\frac{\epsilon\varphi}{e} \right)^3$$

- Seek solution in the form

$$\frac{\epsilon\varphi(r)}{e} = \frac{1}{r} \chi \left(\frac{r}{a} \right)$$

- Via Gauss's theorem χ is related to the charge $Q(r)$ within a sphere of radius r :

$$Q(r) = -r^2 \frac{\partial(\epsilon\varphi/e)}{\partial r} = \chi(\ell) - \chi'(\ell), \quad \ell = \ln \frac{r}{a}$$

- Substitution:

$$\chi''(\ell) - \chi'(\ell) = \gamma \chi^3$$

- For $\ell \gg 1$ the second-order derivative is negligible; then $Q=\chi$ or in natural units $q=Qa=\chi a$, and for **arbitrary** screening strength the nonlinear Thomas-Fermi equation acquires the form

$$\frac{dq}{d\ell} = -\frac{2g\alpha}{3\pi} q^3$$

- This is identical to the Gell-Mann-Low (RG) equation for the physical charge in QED reflecting the effects of vacuum polarization! Physics is different.

Properties of the “flow” equation

$$\frac{dq}{d\ell} = -\frac{2g\alpha}{3\pi} q^3, \quad \ell = \ln \frac{r}{a}$$

- Applicable for $r \gg a$ for arbitrary screening strength; in the strong screening regime it is applicable beyond several source radii.
- It exhibits the Landau “zero charge” effect: for any “initial” value of q the system “flows” to zero charge fixed point $q = 0$ as $\ell \rightarrow \infty$. Alternatively for r fixed complete screening is reached in the point source limit $a \rightarrow 0$.

Solution (log accuracy)

$$q^2(r) = \frac{3\pi}{4g\alpha \ln(r/a)}$$

$$\varphi(r) \approx \frac{e}{\epsilon r \sqrt{2\gamma \ln(r/a)}} = \frac{e}{2\epsilon r} \sqrt{\frac{3\pi}{g\alpha^3 \ln(r/a)}}$$

$$n(r) = \frac{\gamma}{4\pi r^3} \frac{1}{[2\gamma \ln(r/a)]^{3/2}} = \frac{1}{16\pi r^3} \sqrt{\frac{3\pi}{g\alpha^3}} \ln^{-3/2} \left(\frac{r}{a} \right)$$

- The hallmark of the solution is its *near universality* - weak logarithmic dependence on the source size a .

Spherically-symmetric charge distribution: NBGS/QED

- Now we need to look at the equation

$$\frac{dq}{d\ell} = -\frac{2g\alpha}{3\pi} \left(q^2 - \frac{2qr}{\Lambda} \right)^{3/2}, \quad \ell = \ln \frac{r}{a}$$

- $q(r)$ decreases *slower* than its WS counterpart ($\Lambda = \infty$); when the *rhs* = 0 we reach the edge of the space charge region, q acquires its observable value q_∞ and stops changing thereafter.
- The outcome can be understood by “terminating” the WS flow equation at the scale corresponding to the edge of the space charge region $\ell_{sc} = \ln(R_{sc}/a) \gg 1$ and identifying $q(\ell_{sc}) = q_\infty$:

$$q_\infty^2 \approx \frac{3\pi}{4g\alpha \ln(q_\infty \Lambda/a)}$$

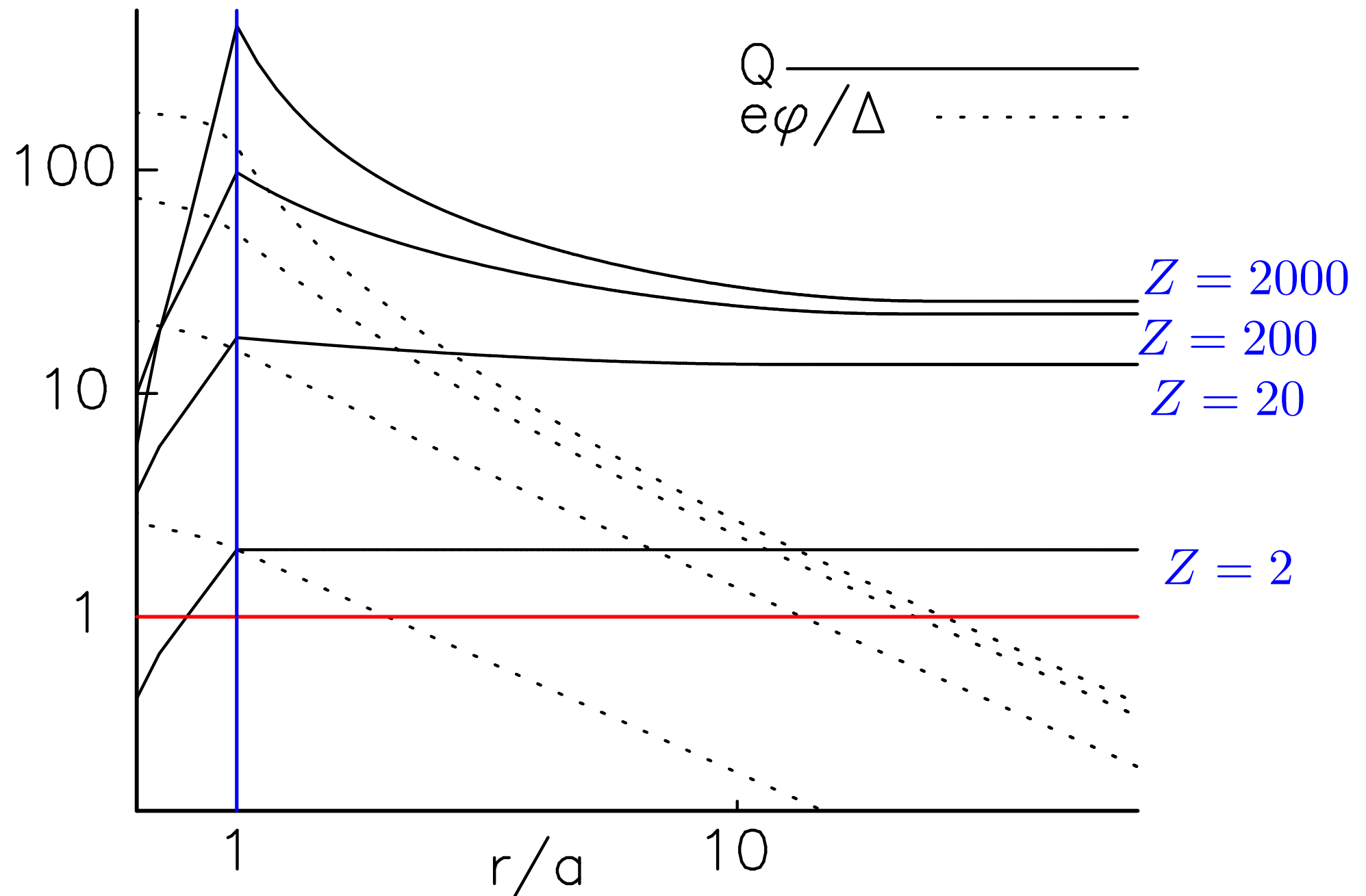
- For a fixed there exists a nearly universal lower limit on the observable charge; in the point source limit there is complete screening.

- Approximation solution

$$q_\infty = Q_\infty \alpha \approx \sqrt{\frac{3\pi}{4g\alpha \ln(\Lambda/a(g\alpha)^{1/2})}}, \quad R_{sc} = \frac{q_\infty \Lambda}{2}$$

- With logarithmic accuracy $Q_\infty \simeq \alpha^{-3/2} \approx 30$ ($Q_\infty \simeq 3000$ in QED) and $R_{sc} \simeq 30nm$.

Numerical solution of the full problem



Confirms the analysis and clearly demonstrates the existence of universal charge in the large Z limit.

Weak screening regime $\gamma Z^2 \ll 1$ and synthesis

- Here the concept of the Debye screening length loses its meaning; the number of the electrons residing within the impurity region (γZ^3) is small; most of them are outside. As the strength of screening increases, more and more of them move inside.
- *Interpolation* solution for the WS case:

$$q^2(r) = \frac{z^2}{1 + (4g\alpha z^2/3\pi) \ln(r/a)} \quad \varphi(r) = \frac{Ze}{\epsilon r \sqrt{1 + (4g\alpha z^2/3\pi) \ln(r/a)}}$$

$$n(r) = \frac{gz^3}{6\pi^2 r^3 [1 + (4g\alpha z^2/3\pi) \ln(r/a)]^{3/2}}$$

These have log accuracy and match both the $r \gg a$ “zero-charge” limit and perturbation theory in γZ^2 . If z is viewed more broadly as the *net* charge within the source region, these equations encompass *all* the regimes of screening.

- Deviations from the Coulomb law become substantial at distances

$$r > R_{scr} \simeq a e^{3\pi/4g\alpha z^2}$$

This is the *screening radius* within the space charge of Weyl electrons. As the strength of screening increases from small to large $\gamma Z^2 \simeq g\alpha z^2$, the screening radius decreases from a very large value to the scale comparable to the source size.

Narrow band-gap semiconductors and QED

- Solution for charge
$$q_\infty^2 \approx \frac{z^2}{1 + (4g\alpha z^2/3\pi) \ln(q_\infty \Lambda/a)}$$
- As $\gamma Z^2 \simeq g\alpha z^2$ increases from small to large values, the initial $q_\infty = z$ growth slows down eventually saturating (for a fixed) at nearly-universal value.
- For realistic $a \propto Z^{1/3}$ and Z large the $Q_\infty(Z)$ dependence is a nearly universal slowly increasing function of Z . This explains (and goes beyond) numerical data of Greiner et al.
- Unfortunately not all is well... Solve for the bare charge z :
$$z^2 \approx \frac{q_\infty^2}{1 - (4g\alpha q_\infty^2/3\pi) \ln(q_\infty \Lambda/a)}$$
- For q_∞ fixed the denominator vanishes for finite a given by
$$a_p \simeq \Lambda q_\infty e^{-3\pi/4g\alpha q_\infty^2}$$
- For $a < a_p$ the bare charge is imaginary! This is certainly unacceptable. This is the “Landau pole” familiar from QED. The Landau pole is the direct consequence of the Landau zero charge. Even though the reality of zero charge in QED is still debatable (I think!), here it is clearly an artifact...

Range of applicability of the Thomas-Fermi theory

- The observable charge q_∞ is the critical charge of the single-particle problem for a charged region of scale R_{sc} which is due to both the external charge and that of the space charge. Then dimensional analysis dictates:

$$q_\infty = f\left(\frac{R_{sc}}{\Lambda}\right), \quad f(y \rightarrow 0) \rightarrow 1, \quad f(y \rightarrow \infty) \simeq y$$

- Only in the classical limit $R_{sc} \gg \Lambda$ does this agree with Thomas-Fermi result!
- For the WS case, $\Lambda = \infty$, the semiclassical condition can never be met!
- Prediction of complete screening in WS is an artifact of the Thomas-Fermi approximation; observable charge in the WS case is $q_\infty = 1$ or, in physical units, $Q_\infty = 1/\alpha$, the inverse fine structure constant.
- Thomas-Fermi analysis is applicable provided $q_\infty \gg 1$ ($Q_\infty \gg 1/\alpha$).
- Prediction of nearly universal observable charge in the NBGS/QED cases survives as $Q_\infty \simeq 1/\alpha^{3/2}$ and $\alpha \ll 1$ (this would not work in graphene).
- The size of the space charge region in the WS case can be estimated by “terminating” the “zero charge” solution at the scale corresponding to $q \simeq 1$:

$$q^2(r) = 3\pi/4g\alpha \ln(r/a) \simeq 1 \rightarrow R_{sc} \simeq ae^{\text{const}/g\alpha}$$
- This Weyl ion could be detectable in a large g material.

Conclusions

- Large field QED effect such as vacuum condensation can be observed in CM setting (in narrow band-gap semiconductors) where required charges are moderate and readily achievable. *Here the new effect of nearly universal observable charge is predicted.*
- Designing controllable experiment to see these effects remains challenging. One possibility is spectroscopy of electron and hole drops in completely compensated NBGS where large charges are also formed (Shklovskii & Efros, 1972).
- Weyl semimetals realize massless version of QED. Here the Thomas-Fermi prediction of complete screening is actually incorrect. This flaw however does not affect the prediction of nearly universal charge in the NBGS case. *The observable charge in the WS case is always given by the inverse of material's fine structure constant.*
- Thomas-Fermi theory overlooks the critical charge effect. But there is a simple physics motivated fix. This improved theory predicts that the vacuum condensation (in the WS case) is a Kosterlitz-Thouless transition (whether this is true remains to be seen).

TABLE I. Summary of properties of electrons in vacua of quantum electrodynamics (QED), narrow band-gap semiconductors (NBGS), and Weyl semimetals (WS).

Media	QED	NBGS	WS
Electrons	free	band Dirac	band Weyl
Mass	m_e	$m \simeq 10^{-2}m_e$	0
Degeneracy g	2	≥ 2	≥ 2
Dielectric constant	$\epsilon = 1$	$\epsilon \approx 10$	$\epsilon \simeq 10$
Limiting speed	c	$v \approx 4 \times 10^{-3}c$	$v \simeq 10^{-2}c$
Band gap or rest energy	$2m_e c^2$	$10^{-7} \times 2m_e c^2$	0
Fine structure constant α	$\frac{e^2}{\hbar c} \approx \frac{1}{137}$	$\frac{e^2}{\hbar v \epsilon} \approx \frac{1}{6}$	$\frac{e^2}{\hbar v \epsilon} \simeq 0.1$
Coupling constant γ	$\frac{4\alpha^3}{3\pi} \approx 10^{-7}$	$\frac{2g\alpha^3}{3\pi} \lesssim 10^{-3}$	$\frac{2g\alpha^3}{3\pi} \lesssim 10^{-3}$
Classical radius of electron	$r_e \simeq 10^{-6} \text{ nm}$	$R_e \simeq 1 \text{ nm}$	∞
Compton wavelength	$\lambda \simeq 10^{-4} \text{ nm}$	$\Lambda \simeq 10 \text{ nm}$	∞
Schwinger or Zener field	$E_S \simeq 10^{16} \frac{\text{V}}{\text{cm}}$	$E_Z \simeq 10^5 \frac{\text{V}}{\text{cm}}$	0

How to improve the accuracy of the Thomas-Fermi analysis in the WS case (speculation)

- Thomas-Fermi theory (naturally) overlooks the critical charge effect. But there is a simple fix...

- Compare

$$p_F^2 = \frac{1}{v^2} ((e\varphi)^2 - e\varphi\Delta) \quad vs \quad p_r^2 = \frac{1}{c^2} (U^2(r) + 2m_e c^2 U(r)) - \frac{M^2}{r^2}$$

- Quantum-mechanical fall to the center is missing but easy to reintroduce (by hand). This modifies the “flow” equation into

$$\frac{dq}{d\ell} = -\frac{2g\alpha}{3\pi} (q^2 - 1)^{3/2}$$

- This resolves difficulties of “zero charge”, preserves Thomas-Fermi findings and also predicts that the vacuum instability is a Kosterlitz-Thouless type transition:

$$R_{sc} = a \exp \left(\frac{3\pi}{2g\alpha} \frac{z}{\sqrt{z^2 - 1}} \right)$$